A NOVEL ALTERNATING DIRECTION IMPLICIT METHOD FOR SOLVING INTERFACE PROBLEMS

Presented by:

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- What are Interface problems?
- What is the Matched Alternating Direction Implicit Method?
- What are our results with this method thus far?
- How have we improved this method?

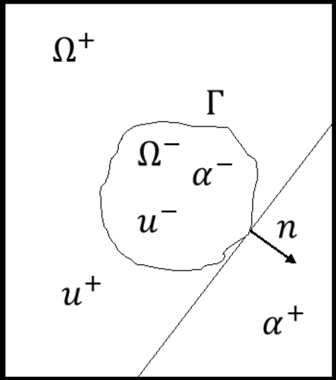
WHAT IS AN INTERFACE PROBLEM?

$$\frac{\partial u}{\partial t} = \nabla \cdot (\alpha \nabla u) + f \text{ in } \Omega = \Omega^- \cup \Omega^+ \qquad (1)$$

Jump Conditions:

$$[u] = u^+ - u^- = \phi(s,t), [\alpha u_n] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{\partial u^-}{\partial n} = \psi(s,t) \qquad (2)$$

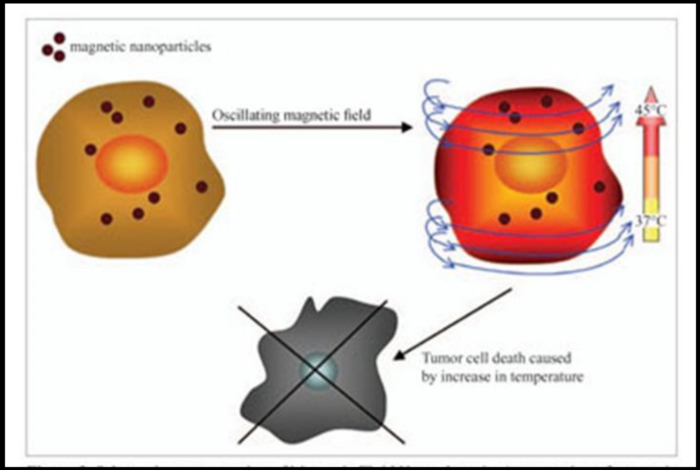
α- diffusion coefficient
u- function of interest
f- source term
Γ- interface
Ω^-- inside of interface
Ω^+- outside of interface
n- normal direction to tangent line

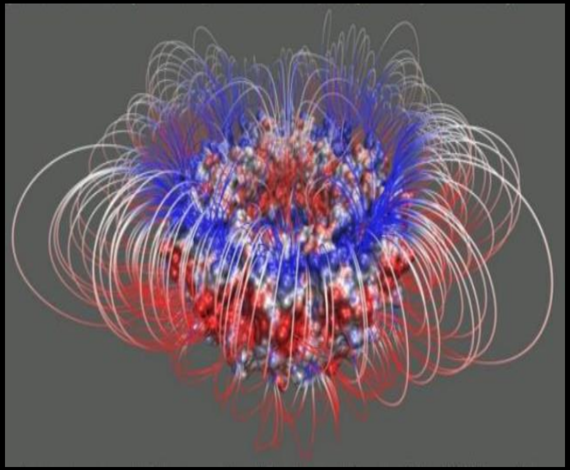


APPLICATIONS

- Penne's Bioheat Equation
- Revolutionary Cancer Treatment

- Poisson-Boltzmann Equation
- Models Electrostatic Potential on Molecular Surfaces





A MATCHED ALTERNATE DIRECTION IMPLICIT METHOD

- Temporal Discretization
  - Douglas Alternating Direction Implicit Method (D-ADI) and Peaceman-Rachford Method
- Spatial Discretization
  - Matched Interface and Boundary Method (MIB)

TEMPORAL DISCRETIZATION

Using the Douglas ADI Method:

- Similar to Implicit Euler Method
- 1st Order Accuracy

$$\left(\frac{1}{\alpha}-\Delta t\delta_{xx}\right)u_{i,j}^{*}=\left(\frac{1}{\alpha}+\Delta t\delta_{yy}\right)u_{i,j}^{k}+\frac{\Delta t}{\alpha}f_{i,j}^{k+1}$$
$$\left(\frac{1}{\alpha}-\Delta t\delta_{yy}\right)u_{i,j}^{k+1}=\frac{1}{\alpha}u_{i,j}^{*}-\Delta t\delta_{yy}u_{i,j}^{k} \tag{3}$$

Using the Peaceman-Rachford Method:

- Similar to Crank-Nicholson Method
- 2nd Order Accuracy

$$\left(\frac{1}{\alpha}-\frac{\Delta t}{2}\delta_{xx}\right)u_{i,j}^{*}=\left(\frac{1}{\alpha}+\frac{\Delta t}{2}\delta_{yy}\right)u_{i,j}^{k}+\frac{\Delta t}{2\alpha}f_{i,j}^{k+\frac{1}{2}}$$
$$\left(\frac{1}{\alpha}-\frac{\Delta t}{2}\delta_{yy}\right)u_{i,j}^{k+1}=\left(\frac{1}{\alpha}+\frac{\Delta t}{2}\delta_{xx}\right)u_{i,j}^{*}+\frac{\Delta t}{2\alpha}f_{i,j}^{k+\frac{1}{2}} \tag{4}$$

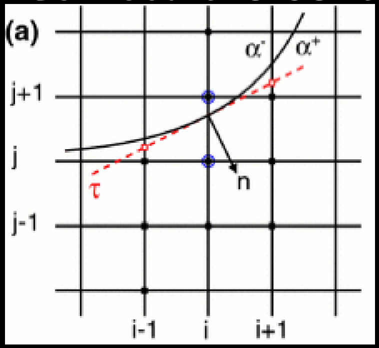
α- diffusion coefficient
Δt- change in time
δ_{xx}- finite difference operator x
δ_{yy}-finite difference operator y
k –current time step
f- source term
u - function of interest

SPATIAL DISCRETIZATION

Use the standard central difference formula on grids away from the interface:

$$\delta_{yy}u_{i,j}^k := \frac{1}{h^2}\left(u_{i,j-1}^k - 2u_{i,j}^k + u_{i,j+1}^k\right) \quad (5)$$

Correct the central difference formula on grids close to the interface:



$$\delta_{yy}u_{i,j}^k := \frac{1}{h^2}\left(\tilde{u}_{i,j-1}^k - 2u_{i,j}^k + u_{i,j+1}^k\right) \quad (6)$$

$$\delta_{yy}u_{i,j}^k := \frac{1}{h^2}\left(u_{i,j-1}^k - 2u_{i,j}^k + \tilde{u}_{i,j+1}^k\right) \quad (7)$$

(8)

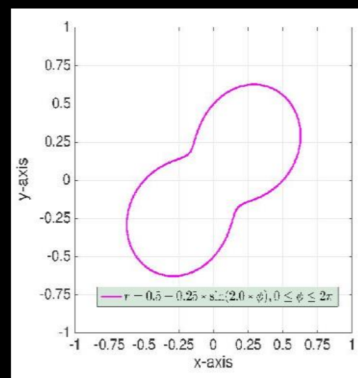
Incorporate the derived jump conditions:

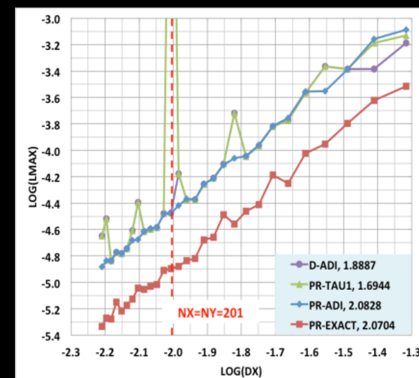
$$[\alpha u_y] = \psi sin\theta + cos\theta(\alpha^+ - \alpha^-)\boldsymbol{u_\tau^+} - cos\theta[\alpha^-\phi_\tau] := \hat{\psi} \quad (9)$$

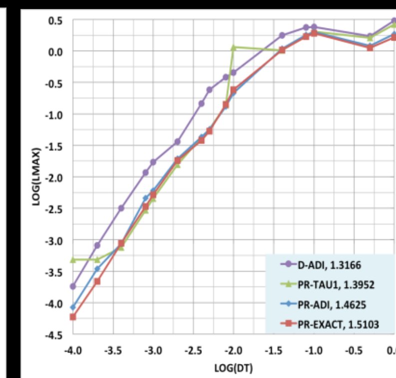
$$[\alpha u_x] = \psi cos\theta - sin\theta(\alpha^+ - \alpha^-)\boldsymbol{u_\tau^+} - sin\theta[\alpha^-\phi_\tau] := \bar{\psi}$$

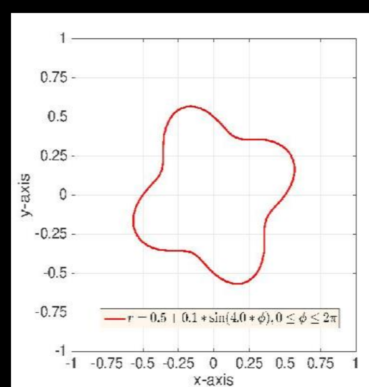
NUMERICAL EXPERIMENTS

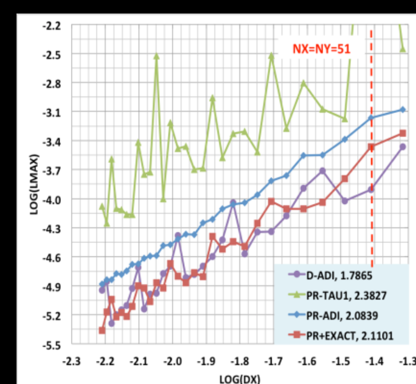
- Accuracy tests for 2-headed and 4-leaf interface examples
- Compare the effectiveness of Douglas ADI, Peaceman-Rachford, and Peaceman-Rachford ADI methods to a given solution

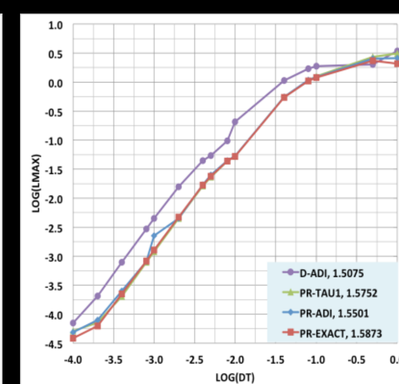




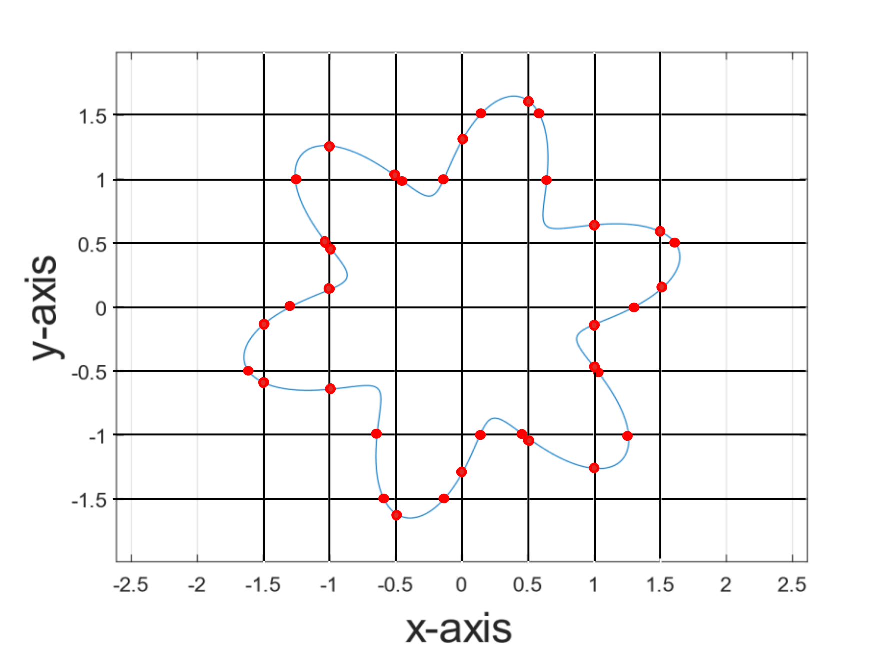


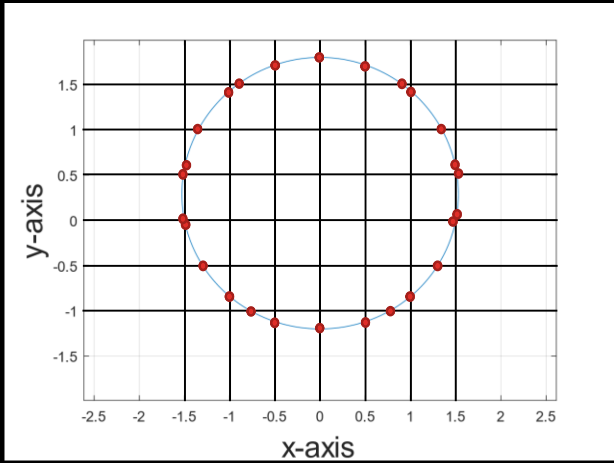






RECENT IMPROVEMENTS
IMPROVED CODING TECHNIQUES

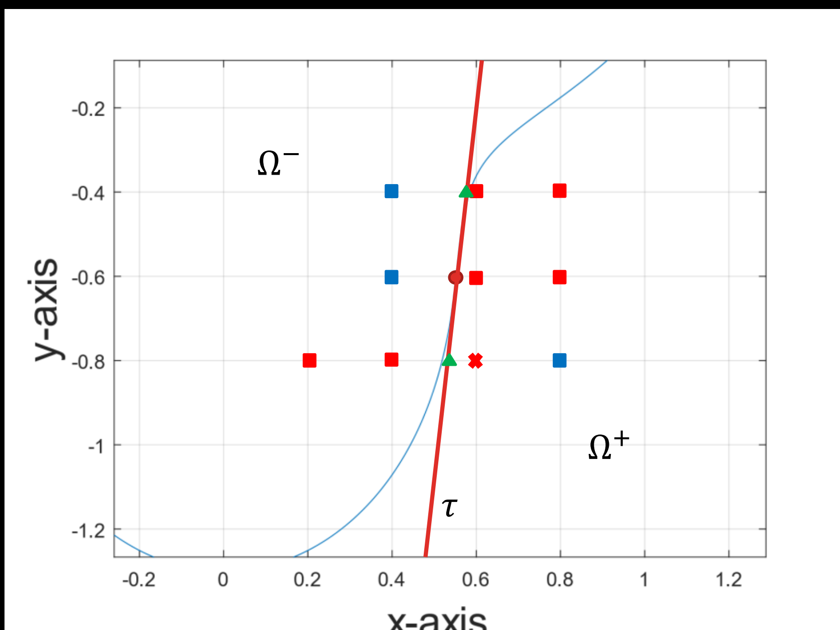




- Can store arbitrary number of Interface Points
- Same speed as before
- Allows for more complicated interface

RECENT IMPROVEMENTS

IMPROVED NUMERICAL METHODS



- Interface points at inflection points can be calculated
- The closest points do not need to be used for interpolation
- Ideally the auxiliary point is roughly halfway between the interpolating points
- $[\alpha u_y] = \psi sin\theta + cos\theta(\alpha^+ - \alpha^-)\boldsymbol{u_\tau^+} - cos\theta[\alpha^- \phi_\tau] := \hat{\psi}$

- Interface Point
- Auxiliary Point
- Interpolating Point
- Interpolating Point (Fictitious)
- Tangent Line
- Interpolating Point (old)

CONCLUSION

- Future improvements
- Overcomes mathematical obstacles in a number of revolutionary applications

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