An Enhanced Augmented Matched Interface and Boundary (AMIB) Method for Solving Elliptic and Parabolic Problems on Irregular 2D Domains

Benjamin Pentecost
bp957721@wcupa.edu

Department of Mathematics
West Chester University of Pennsylvania

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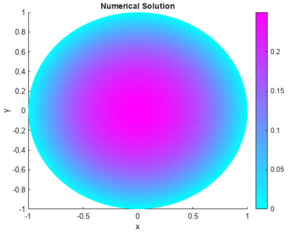
Mathematical Models

- **Poisson Eqn. (time-indept.)**:

$$\Delta u + ku = f(\vec{x}), \qquad (1.1)$$

- Boundary Condition:

$$\alpha_\Gamma u + \beta_\Gamma \frac{\partial u}{\partial n} = \phi(\vec{x}), \qquad (1.2)$$



- **Heat Eqn. (time-dept.)**:

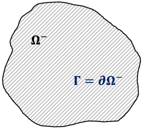
$$\frac{\partial u}{\partial t} = \beta \Delta u + g, \quad 0 \le t \le T, \qquad (1.3)$$

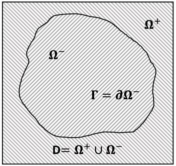
- Boundary Condition:

$$\alpha_\Gamma u + \beta_\Gamma \frac{\partial u}{\partial n} = \psi(t, \vec{x}), \text{ on } \Gamma, \qquad (1.4)$$

- Initial Condition:

$$u(0, \vec{x}) = u_0(\vec{x}), \qquad (1.5)$$





Applications

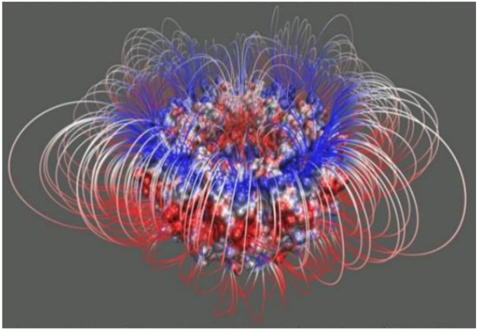


Figure: **Poisson–Boltzmann eqn.** for electrostatic potential distribution over a protein.

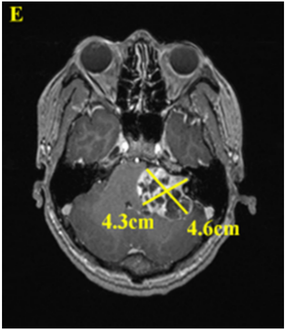
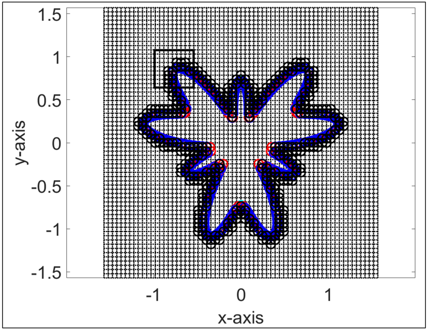
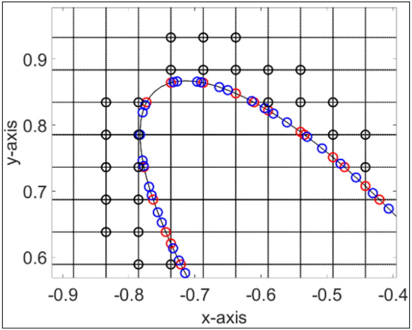


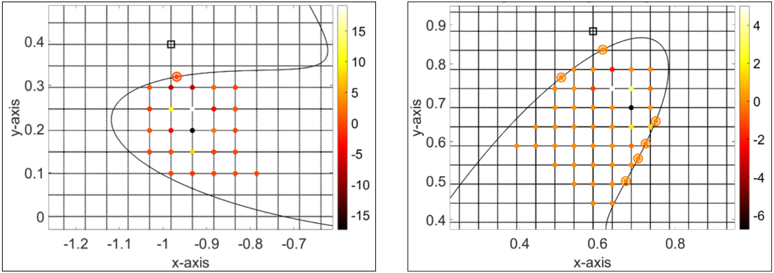
Figure: **Pennes Bioheat eqn.** for heat dissipation in Magnetic Fluid Hyperthermia (MFH).

Interface Points, Fictitious Points, and Vertical Points





Fictitious Value Representations at Fictitious Points



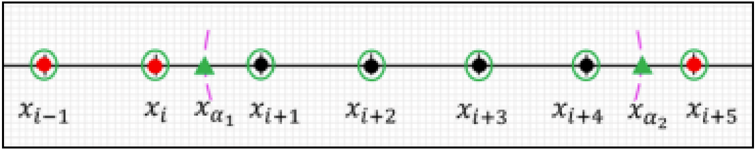
$$\tilde{u}_{\mathtt{FP}} = \sum_{(x_I, y_J) \in S_{\mathtt{FP}}} \breve{w}_{\mathtt{I,J}} u_{\mathtt{I,J}} + \sum_{\vec{x}_{\mathtt{VP_i}} \in V_{\mathtt{FP}}} \breve{w}_{\mathtt{VP_i}} \phi(\vec{x}_{\mathtt{VP_i}}), \quad (2.1)$$

where $S_{\mathtt{FP}}$ is a set of chosen grid points and $V_{\mathtt{FP}}$ is a set of vertical points.

Approximating the Laplacian at Each Interior Gridpoint

$$\delta_{xx}u_{i,j} = \frac{1}{h^2}\left(-\frac{1}{12}u_{i-2,j} + \frac{4}{3}u_{i-1,j} - \frac{5}{2}u_{i,j} + \frac{4}{3}u_{i+1,j} - \frac{1}{12}u_{i+2,j}\right), \tag{2.2}$$

at a grid point (x_i, y_j).



Discretization and Interpolation

$$AU + BQ = F, \qquad (2.3)$$

$$CU + IQ = \Phi, \qquad (2.4)$$

Known

Matrices

$\boldsymbol{A}$ = matrix for solving standard approximation
$\boldsymbol{B}$ = correction term coefficients
$\boldsymbol{C}$ = weights
$\boldsymbol{I}$ = identity matrix

Vectors

$\boldsymbol{F}$ = all $f(x_i)$
$\boldsymbol{\phi}$ = known interface values

Unknown

Vectors

$\boldsymbol{U}$ = all $u(x_i)$
$\boldsymbol{Q}$ = corrections at interfaces

The Augmented System

$$\begin{pmatrix} A & B \\ C & I \end{pmatrix} \begin{pmatrix} U \\ Q \end{pmatrix} = \begin{pmatrix} F \\ \Phi \end{pmatrix}, \tag{2.5}$$

Let N_1 = number of interior grid points, N_2 = number of interface points, we have:

- $A_{N_1 \times N_1}$
- $B_{5N_2 \times N_1}$
- $C_{N_1 \times 5N_2}$
- $I_{5N_2 \times 5N_2}$
- $U_{N_1 \times 1}$
- $Q_{5N_2 \times 1}$

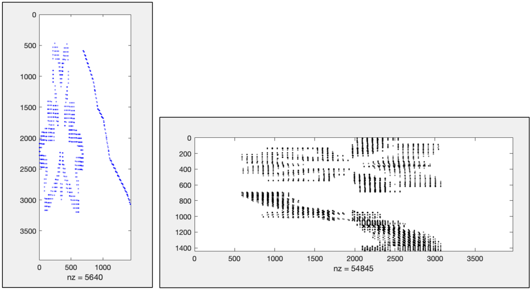


Figure: Nonzero entries of B and C.

The "starfish" Interface (Poisson Eqn.)

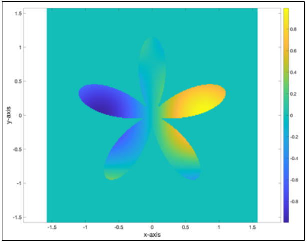


Figure: Numerical solution of the "starfish" interface.

$[N_x, N_y]$	L^∞ error	L^∞ order	L^2 error	L^2 order	BCG iter no.
$[65, 65]$	1.91E-06		6.11E-07		37
$[129, 129]$	1.19E-07	4.00	4.55E-08	3.75	44
$[257, 257]$	5.01E-09	4.57	9.94E-10	5.52	47
$[513, 513]$	2.86E-10	4.13	5.63E-11	4.14	51

The "butterfly" Interface (Heat Eqn.)

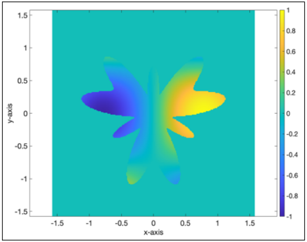


Figure: Numerical solution of the "butterfly" interface.

$[N_x, N_y]$	L^∞		L^2		BCG
	error	order	error	order	time (sec)
[65, 65]	1.15E-04		1.09E-05		28
[129, 129]	5.39E-07	7.74	1.24E-07	6.46	69
[257, 257]	5.50E-09	6.62	1.38E-09	6.48	293
[513, 513]	3.09E-10	4.15	1.02E-10	3.76	1351

Table: Temporal convergence tests for solving the ImIBVP with the "butterfly"-shaped interface

N_t	L^∞		L^2		BCG
	error	order	error	order	time (sec)
2	1.67E-03		9.24E-04		82
4	4.01E-04	2.05	2.23E-04	2.05	160
8	9.99E-05	2.01	5.54E-05	2.01	308
16	2.49E-05	2.00	1.38E-05	2.00	568
32	6.23E-06	2.00	3.46E-06	2.00	1104
64	1.56E-06	2.00	8.65E-07	2.00	2038
128	3.89E-07	2.00	2.16E-07	2.00	3802

The "aircraft" Interface (Heat Eqn.)

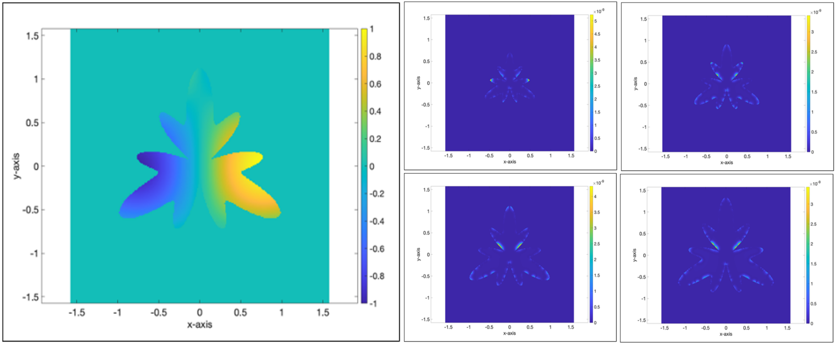


Table: Convergence tests for solving the ImIBVP with the "aircraft"-shaped interface of various scale factors

scale factor k	no. of points IP	no. of points FP	L^∞	L^2	BCG time (sec)
1.0	662	909	5.24E-09	3.78E-10	121
1.3	856	1198	3.41E-09	2.48E-10	122
1.6	1060	1479	4.32E-09	3.16E-10	141
1.9	1266	1765	3.43E-09	2.20E-10	131

Conclusion

Key characteristics of the developed AMIB method are:

- capable of solving problems over highly irregular domains
- capable of handling versatile boundary conditions
- unconditionally stable when solving time-dependent problems
- accelerated by the FFT for high efficiency
- fourth-order accuracy (in space)

References

Li, C., Zhao, S., Pentecost, B., Ren, Y., & Guan, Z. (2024). A fourth-order Cartesian grid method with FFT acceleration for elliptic and parabolic problems on irregular domains and arbitrarily curved boundaries. Submitted for publication.

Li, C., Ren, Y., Long, G., Boerman, E., & Zhao, S. (2023). A Fast Sine Transform Accelerated High-Order Finite Difference Method for Parabolic Problems over Irregular Domains. Journal of Scientific Computing, 95(2), 49-. https://doi.org/10.1007/s10915-023-02177-7

Ren, Y., Feng, H., & Zhao, S. (2022). A FFT accelerated high order finite difference method for elliptic boundary value problems over irregular domains. Journal of Computational Physics, 448, 110762-. https://doi.org/10.1016/j.jcp.2021.110762