A NATURAL PSEUDOMETRIC ON HOMOTOPY GROUPS (2014)

JEREMY BRAZAS AND PAUL FABEL

Abstract. For a path-connected metric space X, the n-th homotopy group $\pi_n(X, x_0)$ inherits a natural pseudometric from the n-th iterated loop space with the uniform metric. When X is compact, the induced pseudometric topology is independent of the original choice of metric on X. In the case of the fundamental group ($n = 1$), the induced pseudometric topology agrees with previously studied topological structures on $\pi_1(X, x_0)$ which are closely related to covering space theory and shape theory.

1. Introduction

Topologies, metrics, and other sturcutres on homotopy groups can be used to retain information about a space that is not detected by the purely algebraic structure. For example, the natural quotient topology on $\pi_1(X, x_0)$ retains a great deal of information about the covering spaces [19], semicovering spaces [3], and other generalized covering spaces [14] over X even when X does not admit a traditional universal covering. In this note[1], we explore a natural approach to defining distance between homotopy classes of maps in metric spaces.

Throughout, let (X, d) be a path-connected metric space and $x_0 \in X$. Let $\Omega^n(X, x_0)$ be the space of maps $\alpha : ([0, 1]^n, \partial[0, 1]^n) \to (X, x_0)$ based at x_0 with the metric of uniform convergence. Let $\alpha^-(t_1, t_2, \ldots, t_n) = \alpha(1 - t_1, t_2, \ldots, t_n)$ be the reverse of α and if $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a sequence of loops, then $\alpha_1 \cdot \alpha_2 \cdots \alpha_n$ is the usual n-fold concatenation defined as α_i on $\left[\frac{i-1}{n}, \frac{i}{n}\right] \times [0, 1]^{n-1}$.

It is well-known that the uniform metric topology agrees with the usual compact-open topology on $\Omega^n(X, x_0)$. Let $\pi_n(X, x_0)$ denote the usual n-th homotopy group and $\pi : \Omega^n(X, x_0) \to \pi_n(X, x_0)$ denote the canonical map taking a loop α to its homotopy class $\pi(\alpha) = [\alpha]$.

We begin by comparing three previously studied topologies on $\pi_n(X, x_0)$.

1.1. **The quotient topology.** Let $\pi_n^{qtop}(X, x_0)$ denote the n-th homotopy group with the quotient topology with respect to the canonical map $\pi : \Omega^n(X, x_0) \to \pi_n(X, x_0)$. It is known that this topology gives $\pi_n(X, x_0)$ the structure of a *quasitopological group* [8, 2] (in the sense that inversion is continuous and multiplication is continuous in each variable [1]) which can fail to be a topological group [10, 11] even when X is a Peano continuum. The group $\pi_n^{qtop}(X, x_0)$ was previously called *n-th topological homotopy group* [16] before learning of the failure of multiplication to be continuous. The authors have sometimes referred to it the n-th quasitopology homotopy group [6]. Regardless of the name used, the natural quotient topology on homotopy groups often retains a great deal of information about the local structure

[1]This paper was written in 2014 and has remained unpublished simply because we never came back to finish it. The formatting and bibliography were updated on 10/15/2021.

of a space, such as the homotopically path-Hausdorff property [6, 12, 18]. Weaker properties like the subgroup-relative homotopically Hausdorff property [14] (see also [9, 20, 21] for the absolute property) are not completely classified by this topology but are certainly related.

1.2. **The τ-topology.** In [4], it was observed that for any quasitopological group G, there is a finest group topology on the group G, which is coarser than that of G. The resulting topological group is denoted $\tau(G)$. In otherwords, the category of topological groups is a reflective subcategory of the category of quasitopological groups, where τ is the reflection functor. In the case of fundamental groups, the τ-reflection $\pi_n^{\tau}(X,x_0)=\tau(\pi_n^{qtop}(X,x_0))$ is a topological group. While the topology of $\pi_n^{\tau}(X,x_0)$ is coarser than that of $\pi_n^{qtop}(X,x_0)$, it is the finest *group* topology on $\pi_n(X,x_0)$ such that $\pi:\Omega^n(X,x_0)\to\pi_n(X,x_0)$ is continuous. This universal property has provided applications to the general theory of topological groups [5] (in particular, free topological groups).

1.3. **The shape topology.** We refer the reader to [7][17] for a more detailed treatment of the first shape group and the "shape-topology" in the case of the fundamental group. The construction for fundamental groups of based spaces is addressed specifically in [7].

Let $cov(X)$ be the directed set of pairs $(\mathscr{U},U_0)$ where $\mathscr{U}$ is a locally finite open cover of X and U_0 is a distinguished element of $\mathscr{U}$ containing x_0. Note $cov(X)$ is directed by refinement. Given $(\mathscr{U},U_0)\in cov(X)$ let $N(\mathscr{U})$ be the abstract simplicial complex which is the nerve of $\mathscr{U}$. In particular, $\mathscr{U}$ is the vertex set of U and the n vertices $U_1,...,U_n$ span an n-simplex $\Leftrightarrow\bigcap_{i=1}^{n}U_i\neq\emptyset$. The geometric realization $|N(\mathscr{U})|$ is a polyhedron and thus $\pi_n^{qtop}(|N(\mathscr{U})|,U_0)$ is a discrete group.

Given a pair $(\mathscr{V},V_0)$ which refines $(\mathscr{U},U_0)$, a simplicial map $p_{\mathscr{U}\mathscr{V}}:|N(\mathscr{V})|\to|N(\mathscr{U})|$ is constructed by sending a vertex $V\in\mathscr{V}$ to some $U\in\mathscr{U}$ for which $V\subseteq U$ (in particular, V_0 is mapped to U_0) and extending linearly. The map $p_{\mathscr{U}\mathscr{V}}$ is unique up to homotopy and thus induces a unique homomorphism $p_{\mathscr{U}\mathscr{V}*}:\pi_n(|N(\mathscr{V})|,V_0)\to\pi_n(|N(\mathscr{U})|,U_0)$. The inverse system

$$(\pi_n(|N(\mathscr{U})|,U_0),p_{\mathscr{U}\mathscr{V}*},cov(X))$$

of discrete groups is the *fundamental pro-group* and the limit $\check{\pi}_n(X,x_0)$ (topologized with the usual inverse limit topology) is the *n-th shape homotopy group.*

Given a partition of unity $\{\phi_U\}_{U\in\mathscr{U}}$ subordinated to $\mathscr{U}$ and such that $\phi_{U_0}(x_0)=1$, a map $p_{\mathscr{U}}:X\to|N(\mathscr{U})|$ is constructed by taking $\phi_U(x)$ (for $x\in U$, $U\in\mathscr{U}$) to be the barycentric coordinate of $p_{\mathscr{U}}(x)$ corresponding to the vertex U. The induced continuous homomorphism $p_{\mathscr{U}*}:\pi_n(X,x_0)\to\pi_n(|N(\mathscr{U})|,U_0)$ satisfies $p_{\mathscr{U}*}\circ p_{\mathscr{U}\mathscr{V}*}=p_{\mathscr{V}*}$ whenever $(\mathscr{V},V_0)$ refines $(\mathscr{U},U_0)$. Thus there is a canonical, continuous homomorphism $\psi:\pi_n(X,x_0)\to\check{\pi}_n(X,x_0)$ to the first shape group.

The *shape topology* on $\pi_n(X,x_0)$ is the initial (or pull-back) topology with respect to the first shape homomorphism $\psi:\pi_n(X,x_0)\to\check{\pi}_n(X,x_0)$. Thus $U\subset\pi_n(X,x_0)$ is open $\Leftrightarrow U=\psi^{-1}(V)$ for an open set $V\subset\check{\pi}_n(X,x_0)$. It is easy to check that the shape topology gives $\pi_n(X,x_0)$ the structure of a topological group.

Proposition 1. [4] *The shape topology of $\pi_n^{shape}(X,x_0)$ is coarser than that of $\pi_n^{\tau}(X,x_0)$.*

Definition 2. The space X is π_n*-shape injective* if $\psi:\pi_n(X,x_0)\to\check{\pi}_n(X,x_0)$ is a monomorphism.

Clearly $\pi_n^{shape}(X,x_0)$ is Hausdorff $\Leftrightarrow$ X is π_n-shape injective.

2. A pseudometric on the n-th homotopy group

Let (X,d) be a path-connected metric space and give $\Omega^n(X,x_0)$ the uniform metric $\mu(\alpha,\beta)=\sup_{t\in[0,1]^n}\{d(\alpha(t),\beta(t))\}$. Observe that

$$\mu(\alpha\cdot\alpha',\beta\cdot\beta')=\max\{\mu(\alpha,\beta),\mu(\alpha',\beta')\}$$

and $\mu(\alpha,\beta)=\mu(\alpha^-,\beta^-)$.

We consider the following function $\rho:\pi_n(X,x_0)\times\pi_n(X,x_0)\to[0,\infty)$ on the homotopy group $\pi_n(X,x_0)$:

$$\rho([\alpha],[\beta])=\inf\{\mu(\alpha,\beta)\mid\alpha\in[\alpha],\beta\in[\beta]\}.$$

We claim that ρ is a pseudometric on $\pi_n(X,x_0)$. Certinaly ρ is symmetric and $\rho([\alpha],[\alpha])=0$, however, it requires a little more work to verify the triangle inequality. Our proof of the triangle inequality applies to all $n\geq 1$ and thus we do not treat the abelian case ($n\geq 2$) separately.

Remark 3. In general, if X is a metric space, $\sim$ is an equivalence relation, and $Y=X/{\sim}$ (where $[a]$ denotes the class of $a\in X$), the definition $\rho:Y\times Y\to[0,\infty)$, $\rho([a],[b])=\inf\{d(a,b)\mid a\in[a],b\in[b]\}$ need not satisfy the triangle inequality. Thus in our situation, we must make use of the group structure of $\pi_n(X,x_0)$ and the nature of the uniform metric μ in order to verify the triangle inequality.

Lemma 4. *Inversion $[\alpha]\mapsto[\alpha^-]$ is an isometry.*

Proof. Since $\mu(\alpha,\beta)=\mu(\alpha^-,\beta^-)$, it is clear that $\rho([\alpha],[\beta])=\rho([\alpha^-],[\beta^-])$ □

Lemma 5. *All right translations $[\alpha]\mapsto[\alpha\cdot\alpha']$ and left translations $[\alpha]\mapsto[\alpha'\cdot\alpha]$ are isometries.*

Proof. Note that $\mu(\alpha\cdot\alpha',\beta\cdot\alpha')=\mu(\alpha,\beta)$ and thus $\rho([\alpha],[\beta])=\rho([\alpha\cdot\alpha'],[\beta\cdot\alpha'])$. The argument for left translations is the same. □

Lemma 6. *For any maps $\alpha,\beta\in\Omega^n(X,x_0)$, $\rho([\alpha{\cdot}\beta],[c])\leq\max\{\rho([\alpha],[c]),\rho([\beta],[c])\}$.*

Proof. Suppose $[\alpha],[\beta]\in\pi_n(X,x_0)$ and $\epsilon>0$. Find $\alpha_1\in[\alpha]$, $\beta_1\in[\beta]$, and $c_1,c_2\in[c]$ such that $\mu(\alpha_1,c_1)<\rho([\alpha],[c])+\frac{\epsilon}{2}$ and $\mu(\beta_1,c_2)<\rho([\beta],[c])+\frac{\epsilon}{2}$. Note that

$$\begin{aligned}\mu(\alpha_1\cdot\beta_1,c_1\cdot c_2) &= \max\{\mu(\alpha_1,c_1),\mu(\beta_1,c_2)\}\\ &< \max\left\{\rho([\alpha],[c])+\frac{\epsilon}{2},\rho([\beta],[c])+\frac{\epsilon}{2}\right\}\\ &< \max\{\rho([\alpha],[c]),\rho([\beta],[c])\}+\epsilon\end{aligned}$$

Thus $\rho([\alpha\cdot\beta],[c])\leq\max\{\rho([\alpha],[c]),\rho([\beta],[c])\}$. □

Proposition 7. *ρ is a pseudometric on $\pi_n(X,x_0)$.*

Proof. As noted above, it suffices to verify the triangle inequality. Let $[\alpha],[\beta],[\gamma]\in\pi_n(X,x_0)$. Using the previous three lemmas, we have:

$$\begin{aligned}\rho([\alpha],[\gamma]) &= \rho([\alpha\cdot\gamma^-],[c])\\ &= \rho([\alpha\cdot\beta^-\cdot\beta\cdot\gamma^-],[c])\\ &\leq \max\{\rho([\alpha\cdot\beta^-],[c]),\rho([\beta\cdot\gamma^-],[c])\}\\ &= \max\{\rho([\alpha\cdot\beta^-],[c]),\rho([\gamma\cdot\beta^-],[c])\}\\ &\leq \rho([\alpha\cdot\beta^-],[c])+\rho([\gamma\cdot\beta^-],[c])\\ &= \rho([\alpha],[\beta])+\rho([\gamma],[\beta])\\ &= \rho([\alpha],[\beta])+\rho([\beta],[\gamma])\end{aligned}$$

□

Theorem 8. *Equipped with the topology induced by the pseudometric ρ, $\pi_n(X,x_0)$ is a topological group whose open balls $B_\rho([c],r)$ are open normal subgroups.*

Proof. Since the open balls $B_\rho([c],r)$ form a neighborhood base at $[c]$, it will follow that $\pi_n(X,x_0)$ is a topological group once we show that $B_\rho([c],r)$ is an open subgroup.

Since $\rho([\alpha],[c])=\rho([\alpha^-],[c])$, $B_\rho([c],r)$ is closed under inversion. Additionally,

$$\rho([\gamma\cdot\alpha\cdot\gamma^-],[c])=\rho([\alpha],[\gamma^-\cdot\gamma])=\rho([\alpha],[c])$$

for any $[\gamma]\in\pi_n(X,x_0)$. Thus $B_\rho([c],r)$ is closed under conjugation (particularly when $n=1$). Finally, suppose $\rho([\alpha],[c]),\rho([\beta],[c])<r$. Then $\rho([\alpha\cdot\beta],[c])\leq\max\{\rho([\alpha],[c]),\rho([\beta],[c])\}<r$ by Lemma 6 and it follows that $B_\rho([c],r)$ is closed under multiplication. □

Let $\pi_n^{met}(X,x_0)$ denote the group with topology induced by the pseudometric ρ. We call this topology simply the **pseudometric topology**. Since translations and inversion in $\pi_n^{met}(X,x_0)$ are isometries and there is a neighborhood base of open normal subgroups at the identity element, it is clear that $\pi_n^{met}(X,x_0)$ is a topological group. Since open subgroups of topological groups are also closed, $\pi_n^{met}(X,x_0)$ is zero-dimensional. On the other hand, $\pi_n^{met}(X,x_0)$ need not be Hausdorff, since the closed normal subgroup $\bigcap_{r>0}B_\rho([c],r)=\{[\alpha]\in\pi_1(X,x_0)\mid\rho([\alpha],[c])=0\}$, is the closure of the identity element and is non-empty if and only if there are sequences of maps $\alpha_k,\beta_k\in\Omega^n(X,x_0)$ such that $[\alpha_k]=[\alpha_{k+1}]$, $[\beta_k]=[\beta_{k+1}]$ and $\lim_{k\to\infty}\mu(\alpha_k,\beta_k)=0$.

The following example illustrates that when X is non-compact the pseudometric on $\pi_n^{met}(X,x_0)$ is dependent on the metric d on X.

Example 9. Consider the cylinder $X=\mathbb{R}\times S^n$ with basepoint $x_0=(1,b)$. It is easy to see that with the natural product metric d_1, the resulting pseudometric ρ_1 on $\pi_1(\mathbb{R}\times S^n,x_0)$ is discrete. For instance, $\pi_n^{met}(S^n,(1,0))$ is discrete and we may then apply Proposition 10 to the projection (and homotopy equivalence) $\mathbb{R}\times S^n\to S^n$.

On the other hand, we may give $\mathbb{R}\times S^n$ the metric d_2 of the homeomorphic punctured plane $\mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$. Let $x_0=(1,0,0,...,0)\in\mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$ be the basepoint and ρ_2 denote the resulting pseudometric on $\pi_n(X,x_0)$. Take $\alpha_n:[0,1]\to X$ to be the linear path from x_0 to $(1/n,0,...,0)$ and let γ_n be the embedding of the n-sphere of radius $1/n$ centered at the origin. Now consider the maps $\alpha_n*\gamma_n$

(where $*$ indicates the usual π_1-action on π_n) all of which represent a generator g of $\pi_n(X,x_0)\cong\mathbb{Z}$. We also consider the null-homotopic maps $\alpha_n * c_n$ where $c_n\in\Omega^n(X,x_0)$ is the constant map. But

$$\lim_{n\to\infty}\mu(\alpha_n * c_n,\alpha_n*\gamma_n)=0.$$

Thus $\rho_2(g,1)=0$ and it follows that the resulting pseudometric group is indiscrete.

The previous example shows that in general (non-compact) situations, the pseudometric structure on $\pi_n(X,x_0)$ is not an invariant of homeomorphism type. Rather, it is an invariant of isometry type for metric spaces. Perhaps, there is some geometric relation (weaker than isometry) on metric spaces under which π_n^{met} is invariant. However, the authors have not exactly identified what such a thing could be.

Proposition 10. *If $f:(X,x_0)\to(Y,y_0)$ is a uniformly continuous map, then the induced homomorphism $f_*:\pi_n^{met}(X,x_0)\to\pi_n^{met}(Y,y_0)$ is continuous.*

Proof. Suppose $\epsilon>0$. There is a $\delta>0$ such that $\mu(\alpha,\beta)<\delta\Rightarrow\mu(f\circ\alpha,f\circ\beta)<\epsilon/2$. Suppose $\rho([\alpha],[c])<\delta$. There are $\alpha\in[\alpha]$, $\beta\in[c]$ such that $\mu(\alpha,\beta)<\delta$. Thus $\mu(f\circ\alpha,f\circ\beta)<\epsilon$. It follows that $\rho([f\circ\alpha],[c])=\rho([f\circ\alpha],[f\circ\beta])<\epsilon$. □

Corollary 11. *If X is compact, then the topology of $\pi_n^{met}(X,x_0)$ is independent of the choice of metric on X.*

Proof. Suppose metrics d_1 and d_2 on X induce the same topology. Let ρ_1 and ρ_2 be the respective pseudometrics on $\pi_1^{qtop}(X,x_0)$. Since X is compact, the identity maps $id:(X,d_1)\to(X,d_2)$ and $id:(X,d_2)\to(X,d_1)$ are uniformly continuous. Consequently, the induced homomorphisms on $\pi_n(X,x_0)$ (with the respective pseudometrics) are inverse isomorphisms. □

We also observe that, in general, the isomorphism class of the topological group $\pi_n^{met}(X,x_0)$ does not depend on the choice of basepoint. Recall that for any path $\gamma:[0,1]\to X$, there is a natural map $\Omega^n(X,\gamma(1))\to\Omega^n(X,\gamma(0))$, $\beta\mapsto\gamma*\beta$ which induces a change of basepoint isomorphism $\Gamma:\pi_n(X,\gamma(1))\to\pi_n(X,\gamma(0))$. When γ is a loop based at x_0, these maps induces the usual action of $\pi_1(X,x_0)$ on $\pi_n(X,x_0)$. In the case that $n=1$, this action is the action of $\pi_1(X,x_0)$ on itself by conjugation.

Proposition 12. *For any path $\gamma:[0,1]\to X$, the group isomorphism*

$$\Gamma:\pi_n^{met}(X,\gamma(1))\to\pi_n^{met}(X,\gamma(0)),$$

$\Gamma([\beta])=[\gamma*\beta]$ is an isometry.

Proof. For all $\alpha,\beta\in\Omega^n(X,\gamma(1))$, we have $\mu(\alpha,\beta)=\mu(\gamma*\alpha,\gamma*\beta)$ and thus

$$\rho([\gamma*\alpha],[\gamma*\beta])\leq\rho([\alpha],[\beta]).$$

Thus Γ is non-expansive. The inverse $\Gamma^{-1}:\pi_1^{met}(X,\gamma(0))\to\pi_1^{met}(X,\gamma(1))$, $\Gamma^{-1}([\beta])=[\gamma^-*\beta]$ is non-expansive for the same reason (replacing γ with γ^-). Thus Γ is an isometry. □

Corollary 13. *The fundamental group $\pi_1(X,x_0)$ acts on $\pi_n^{met}(X,x_0)$ by isometry.*

Finally, we compare the pseudometric topology to the quotient and τ-topologies.

Proposition 14. *The function $\pi : \Omega(X,x_0) \to \pi_1^{met}(X,x_0)$, $\pi(\alpha) = [\alpha]$ is continuous. Thus the topology of both $\pi_1^{qtop}(X,x_0)$ and $\pi_1^{\tau}(X,x_0)$ is finer than that of $\pi_1^{met}(X,x_0)$.*

Proof. Suppose $\alpha_n \to \alpha$ in $\Omega(X,x_0)$ and $\epsilon > 0$. There is an N such that $\mu(\alpha,\alpha_n) < \epsilon$ for $n \geq N$. Thus $\rho([\alpha],[\alpha_n]) < \epsilon$ for $n \geq N$ showing that $[\alpha_n] \to [\alpha]$. □

2.1. **Compact metric spaces.** In this section, we consider the pseudometric topology of $\pi_1^{met}(X,x_0)$ when the metric space X is compact.

Lemma 15. *If X is compact, locally path-connected, and semi-locally simply connected, then $\pi_1^{met}(X,x_0)$ is discrete.*

Proof. □

Example 16. If X is a finite polyhedron or a compact manifold, then $\pi_1^{met}(X,x_0)$ is discrete.

Lemma 17. *If X is compact, then the pseudometric topology of $\pi_1^{met}(X,x_0)$ is finer than that of $\pi_1^{shape}(X,x_0)$.*

Proof. Since X is compact, we may assume the pro-fundamental group is of the form

$$(\pi_1(X_n,x_n), p_{n,n+1}, \mathbb{N}),$$

that is, indexed by the natural numbers $\mathbb{N}$. Moreover, since X is compact, X_n is a finite (hence compact) polyhedron for each $n \geq 1$. By Lemma 15, $\pi_1^{met}(X_n,x_n)$ is discrete for each $n \geq 1$. Consequently, the inverse limit $\varprojlim_n \pi_1(X_n,x_n)$ is isomorphic to the shape group $\check{\pi}_1(X,x_0)$ with the usual inverse limit topology. Note that since X is compact, the canonical maps $p_n : X \to X_n$ inducing shape map $\psi : \pi_1(X,x_0) \to \check{\pi}_1(X,x_0)$ are uniformly continuous. By Proposition 10, the homomorphisms $(p_n)_* : \pi_1^{met}(X,x_0) \to \pi_1^{met}(X_n,x_0)$ are continuous and thus $\psi : \pi_1^{met}(X,x_0) \to \varprojlim_n \pi_1(X_n,x_n) = \check{\pi}_1(X,x_0)$ is continuous. Since shape topology on $\pi_1(X,x_0)$ is the coarsest topology on $\pi_1(X,x_0)$ such that ψ is continuous, the pseudometric topology must be finer. □

Remark 18. Cerainly the previous lemma fails when X is no longer required to be compact. If $X = \mathbb{R}^2\backslash\{(0,0)\}$ is the punctured plane as in Example 9, then $\pi_1^{met}(X,x_0)$ is the indiscrete group of integers whereas $\pi_1^{shape}(X,x_0)$ is the discrete group of integers. Thus the identity homomorphism $\psi : \pi_1^{met}(X,x_0) \to \pi_1^{shape}(X,x_0)$ is not continuous.

For a general topological space X and open cover $\mathscr{U}$ of X, recall the Spanier group of X with respect to $\mathscr{U}$ is the normal subgroup $\pi^s(\mathscr{U},x_0)$ of $\pi_1(X,x_0)$ generated by elements $[\alpha \cdot \gamma \cdot \alpha^-] \in \pi_1(X,x_0)$ where γ has image in some $U \in \mathscr{U}$ [19]. We recall a few known facts abotu Spanier groups. The first Lemma is proven directly in [15] and also follows from arguments in Section 3.2 of [6].

Lemma 19. [15] *If X is locally path connected, then $\pi^s(\mathscr{U},x_0)$ is open in the quasitopological fundamental group $\pi_1^{qtop}(X,x_0)$. Moreover, for every open normal subgroup $N \subseteq \pi_1^{qtop}(X,x_0)$, there is an open cover $\mathscr{U}$ of X such that $\pi^s(\mathscr{U},x_0) \subseteq N$.*

Lemma 20. [7] *If X is a locally path connected metric space, then the Spanier groups $\pi^s(\mathscr{U}, x_0)$, $\mathscr{U} \in cov(X)$ form a neighborhood base at the identity element in $\pi_1^{shape}(X, x_0)$ (the fundamental group with the shape topology).*

Theorem 21. *If X is a Peano continuum, then the pseudometric topology and shape topology on $\pi_1(X, x_0)$ agree.*

Proof. By Lemma 17, it suffices to show the open balls $B_\rho([c], r)$, $r > 0$ at the identity element are open in the shape topology of $\pi_1^{shape}(X, x_0)$. Recall that by Theorem 8, $B_\rho([c], r)$ is a normal subgroups of $\pi_1(X, x_0)$. Since the quotient topology of $\pi_1^{qtop}(X, x_0)$ is finer than that of $\pi_1^{met}(X, x_0)$ (Proposition 14), $B_\rho([c], r)$ is an open normal subgroup of $\pi_1^{qtop}(X, x_0)$. By Lemma 19, there is an open cover $\mathscr{U}$ of X such that $\pi^s(\mathscr{U}, x_0) \subseteq B_\rho([c], r)$. By Lemma 20, $\pi^s(\mathscr{U}, x_0)$ is open in the shape topology of $\pi_1^{shape}(X, x_0)$. Since $B_\rho([c], r)$ decomposes as a union of the open cosets of $\pi^s(\mathscr{U}, x_0)$, $B_\rho([c], r)$ is open in $\pi_1^{shape}(X, x_0)$. $\square$

Corollary 22. *For a Peano continuum X, the following are equivalent:*

(1) *ρ is a metric,*
(2) *$\pi_1^{met}(X, x_0)$ is T_4,*
(3) *X is π_1-shape injective.*

Proof. (1) $\Rightarrow$ (2) if ρ is a metric, then $\pi_1^{met}(X, x_0)$ is a metrizable topological group and is necessarily T_4. (2) $\Rightarrow$ (3) If $\pi_1^{met}(X, x_0)$ is T_4, then $\pi_1^{met}(X, x_0)$ is Hausdorff. Since $\pi_1^{met}(X, x_0) \cong \pi_1^{shape}(X, x_0)$, it follows that $\psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is injective. (3) $\Rightarrow$ (1) If $\psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is injective, then $\pi_1^{met}(X, x_0) \cong \pi_1^{shape}(X, x_0)$ is Hausdorff. Since the topology of $\pi_1^{met}(X, x_0)$ is generated by the pseudometric ρ, ρ is a metric. $\square$

Another way to look at the previous Corollary is that X fails to be π_1-shape injective $\Leftrightarrow$ there are sequences of loops α_n, β_n with $[\alpha_n] = [\alpha_{n+1}]$, $[\beta_n] = [\beta_{n+1}]$ for all $n \geq 1$, and $\lim\limits_{n\to\infty} \mu(\alpha_n, \beta_n) = 0$ but such that $[\alpha_n] \neq [\beta_n]$.

It is clear from Remark 18 that Theorem 21 and Corollary 22 can fail to hold for non-compact metrix spaces. We also show that the assumption of local path connectedness cannot be removed using an example from [9, 13].

Example 23. Consider the following subsets of $\mathbb{R}^3$ defined using cylindrical coordinates. Let $Y = \left\{(r, \theta, z) \mid 1 < r < 2, z = \sin\left(\frac{1}{r-1}\right)\right\}$ and $Z = \{(r, \theta, z) \mid 0 \leq r \leq 1, -1 \leq z \leq 1\}$. Observe that $Y \cup Z$ can be obtained by rotating the (Cartesian coordinate-defined) space

$$([0, 1] \times \{0\} \times [-1, 1]) \cup \left\{(x, 0, z) \mid 1 < x \leq 2, z = \sin\left(\frac{1}{x-1}\right)\right\}$$

in the xz-axis about the z-axis. Let $A \subset \mathbb{R}^3$ be an arc connected the cylindrical points $(0, 0, 1)$ and $(2, 0, \sin(1))$ and whose interior is disjoint from $Y \cup Z$. Finally, let $X = Y \cup Z \cup A$.

It is easy to see that $\pi_1(X, x_0) \cong \mathbb{Z}$ and $\check{\pi}_1(X, x_0) \cong \mathbb{Z}$, however the shape homomorphism $\psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is constant. Consequently, $\pi_1^{shape}(X, x_0)$ is indiscrete. On the other hand, it is not too difficult to verify that $\pi_1^{met}(X, x_0)$ is discrete.

Finally, we note a connection to covering space theory. If X is locally path connected and metrizable, then the open normal subgroups of $\pi_1^{shape}(X,x_0)$ are classified by the regular covering maps over X [7]. Since the open normal subgroups form a basis at the identity in $\pi_1^{shape}(X,x_0)$ (by definition of this topology), we have the following consequence.

Corollary 24. *Let X be a Peano continnum. Then a subgroup $H \leq \pi_1^{met}(X,x_0)$ is open if and only if there exists a covering map $p:(Y,y_0)\to(X,x_0)$ such that $p_{\#}(\pi_1(Y,y_0))=H$.*

References

[1] A. Arhangel'skii, M. Tkachenko, *Topological Groups and Related Structures*, Series in Pure and Applied Mathematics, Atlantis Studies in Mathematics, 2008.

[2] J. Brazas, *The topological fundamental group and free topological groups*, Topology Appl. 158 (2011) 779–802.

[3] J. Brazas, *Semicoverings: a generalization of covering space theory*, Homology Homotopy Appl. 14 (2012), 33–63.

[4] J. Brazas, *The fundamental group as a topological group*, Topology Appl. 160 (2013) 170–188.

[5] J. Brazas, *Open subgroups of free topological groups*, Fundamenta Mathematicae 226 (2014) 17-40.

[6] J. Brazas, P. Fabel, *On fundamental groups with the quotient topology*, J. Homotopy and Related Structures 10 (2015) 71–91.

[7] J. Brazas, P. Fabel, *Thick Spanier groups and the first shape group*, Rocky Mountain J. Math. 44 (2014) 1415–1444.

[8] J.S. Calcut, J.D. McCarthy, *Discreteness and homogeneity of the topological fundamental group*, Topology Proc. 34 (2009) 339-349.

[9] G. Conner, M. Meilstrup, D. Repovš, A. Zastrow, M. Željko, *On small homotopies of loops*, Topology Appl. 155 (2008) 1089–1097.

[10] P. Fabel, *Multiplication is discontinuous in the Hawaiian earring group*, Bull. Pol. Acad. Sci. Math. 59 (2011) 77–83.

[11] P. Fabel, *Compactly generated quasitopological homotopy groups with discontinuous multiplication*, Topology Proc. 40 (2012) 303–309.

[12] H. Fischer, D. Repovš, Z. Virk, and A. Zastrow, *On semilocally simply connected spaces*, Topology Appl. 158 (2011) 397–408.

[13] H. Fischer, A. Zastrow, *The fundamental groups of subsets of closed surfaces inject into their first shape groups*, Algebraic and Geometric Topology 5 (2005) 1655–1676.

[14] H. Fischer and A. Zastrow, *Generalized universal covering spaces and the shape group*, Fund. Math. 197 (2007) 167–196.

[15] H. Fischer and A. Zastrow, *A core-free semicovering of the Hawaiian Earring*, Preprint, arXiv:1301.3746. 2013.

[16] H. Ghane, Z. Hamed, B. Mashayekhy, H. Mirebrahimi, *On topological homotopy groups of n-Hawaiian like spaces*, Topology Proc., 36 (2010) 255–266.

[17] S. Mardešić and J. Segal, *Shape theory*, North-Holland Publishing Company, 1982.

[18] B. Mashayekhy, A. Pakdaman, H. Torabi, *Spanier spaces and covering theory of non-homotopically path Hausdorff spaces*, Georgian Mathematical Journal, 20 (2013) 303-317.

[19] E. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.

[20] Z. Virk, *Homotopical smallness and closeness*, Topology Appl. 158 (2011) 360–378.

[21] Z. Virk, A. Zastrow *A homotopically Hausdorff space which does not admit a generalized universal covering*, Topology Appl. 160 (2013) 656–666.