Regular coverings, Spanier groups, and a topologized fundamental group

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Abstract

Regular coverings of locally wep-connected spaces (this includes all locally path connected spaces) are classified up to equivalence by open normal subgroups of the fundamental group with the finest group topology such that the canonical map from the loop space identifying homotopy classes is continuous.

The purpose of this brief note is to use a special case of the classification of semicoverings in [2] to classify regular coverings (in the usual sense) of arbitrary locally path connected spaces. In fact, the classification here holds for a larger class of spaces (locally wep-connected spaces introduced in the aforementioned paper) which includes many non-locally path connected spaces. For the sake of brevity, we refer to [2] for definitions, preliminaries, and notation.

1 Regular coverings

Fix, once and for all, a locally wep-connected topological space X. A *regular covering* of X, with fixed basepoint x_0, is a covering map $p : W \to X$ (in the classical sense) such that the image of the induced homomorphism $p_* : \pi_1(W, w_0) \to \pi_1(X, x_0)$ is normal in $\pi_1(X, x_0)$. In general, all covering spaces are assumed to be path connected. Two regular coverings $p : W \to X$, $p' : W' \to X$ of X are equivalent if there is a homeomorphism $h : W \to W'$ such that $p' \circ h = p$. When X is semi-locally 1-connected, the correspondence $p \mapsto Im(p_*)$ induces a bijection between equivalence classes of regular semicoverings and normal subgroups of $\pi_1(X, x_0)$. When spaces with more complicated local properties are considered, this classification of regular coverings no longer holds. For instance, a space may fail to have a universal covering which corresponds to the trivial subgroup. The difficulty then lies in the identification of the normal subgroups which do correspond to regular coverings. This is achieved in the following theorem.

Theorem 1. *Let X be a locally wep-connected space and $x_0 \in X$. There is a canonical bijection between the equivalence classes of regular coverings of X and open normal subgroups of $\pi_1^{\tau}(X, x_0)$.*

The main result in [2] is, Theorem 3.4, the identification of a canonical bijection between equivalence classes of semicoverings of a locally wep-connected space X and conjugacy classes of open subgroups of the topologized fundamental group $\pi_1^\tau(X, x_0)$ of [1]. The topology of $\pi_1^\tau(X, x_0)$ is characterized as the finest group topology on the fundamental group $\pi_1(X, x_0)$ such that $\Omega(X, x_0) \to \pi_1(X, x_0)$, $\alpha \mapsto [\alpha]$ is continuous. This classification can be restricted to a bijection between equivalence classes of semicoverings $p : W \to X$ such that $Im(p_*)$ is normal in $\pi_1(X, x_0)$ and open normal subgroups of $\pi_1^\tau(X, x_0)$. Thus Theorem 1 will follow if every such semicovering is actually a covering of X. This is precisely Proposition 2 below.

Proposition 2. *Let X be locally wep-connected and locally path connected at its basepoint x_0. If $p : W \to X$ is a semicovering such that $H = p_*(\pi_1(W, w_0))$ is normal, then p is a covering map.*

Proof. Since p is equivalent to $p_H : \tilde{X}_H \to X$ (in **SCov(X)**) by the classification of semicoverings, we use the above description of the topology of $\tilde{X}_H$ to show that p_H is a covering map. Suppose, as above, that $x \in X$, $\alpha \in P(X, x_0)$ is locally well-targeted such that $\alpha(1) = x$, $\mathcal{U} = \bigcap_{j=1}^{n} \langle K_n^j, U_j \rangle$ is an open neighborhood of α such that $\mathcal{U}^{[0,\frac{1}{2}]} \cap \left(\mathcal{U}^{-1}\right)^{[\frac{1}{2},1]} \subseteq \pi^{-1}(H)$, and V is an open neighborhood of x chosen so that for each $v \in V$, there is a path $\gamma \in \mathcal{U}$ from x_0 to v. By Lemma 7.8 of [2], p_H maps the open neighborhood $B(\alpha, \mathcal{U}, V)$ of $[\alpha]_H$ homeomorphically onto V. We show that V is evenly covered by p_H. For each $\beta \in \Omega(X, x_0)$, let $\mathcal{V} = \bigcap_{k=1}^{m} \langle K_m^k, B_k \rangle$ be an open neighborhood of β contained in the pullback of the coset, $\pi^{-1}([\beta]H)$ and such that $B_1 = B_m \subseteq U_1$ is path connected. Now

$$\mathcal{W} = \mathcal{V}^{[0,\frac{1}{2}]} \cap \mathcal{U}^{[\frac{1}{2},1]}$$

is an open neighborhood of the locally well-targeted path $\beta * \alpha$ in $P(X, x_0)$. If $v \in V$, then there is a path $\beta * \gamma \in \mathcal{W}$ such that $\beta * \gamma(1) = v$. It now suffices to check that $\mathcal{W}^{[0,\frac{1}{2}]} \cap \left(\mathcal{W}^{-1}\right)^{[\frac{1}{2},1]} \subseteq \pi^{-1}(H)$ for $B(\beta * \alpha, \mathcal{W}, V)$ to be a well-defined open neighborhood of $[\beta * \alpha]_H$. If $\delta \in \mathcal{W}^{[0,\frac{1}{2}]} \cap \left(\mathcal{W}^{-1}\right)^{[\frac{1}{2},1]}$ is a loop, then $\delta\left(\frac{1}{4}\right), \delta\left(\frac{3}{4}\right) \in B_1$. Let ϵ_1, ϵ_2 be a path in B_1 from x_0 to $\delta\left(\frac{1}{4}\right), \delta\left(\frac{3}{4}\right)$ respectively. Since a reparameterization of $\delta_{[0,\frac{1}{4}]} * \epsilon_1^{-1}$ is a loop in $\mathcal{V}$, we have $\left[\delta_{[0,\frac{1}{4}]} * \epsilon_1^{-1}\right] \in [\beta]H$. Similarly, $\mathcal{V}^{-1}$ is an open neighborhood of a reparameterization of $\epsilon_3 * \delta_{[\frac{3}{4},1]}$ and thus $\left[\epsilon_3 * \delta_{[\frac{3}{4},1]}\right] \in H[\beta]^{-1}$. Since ϵ_1 and ϵ_3 have image in $B_1 = B_m \subseteq U_1$, a reparameterization of $\epsilon_1 * \delta_{[\frac{1}{4},\frac{3}{4}]} * \epsilon_3^{-1}$ gives a loop in $\mathcal{U}^{[0,\frac{1}{2}]} \cap \left(\mathcal{U}^{-1}\right)^{[\frac{1}{2},1]} \subseteq \pi^{-1}(H)$. Thus $\left[\epsilon_1 * \delta_{[\frac{1}{4},\frac{3}{4}]} * \epsilon_3^{-1}\right] \in H$. Since H is normal,

$$[\delta] = \left[\delta_{[0,\frac{1}{4}]} * \epsilon_1^{-1}\right]\left[\epsilon_1 * \delta_{[\frac{1}{4},\frac{3}{4}]} * \epsilon_3^{-1}\right]\left[\epsilon_3 * \delta_{[\frac{3}{4},1]}\right] \in [\beta]HHH[\beta]^{-1} = H$$

and thus $\delta \in \pi^{-1}(H)$. Another application of [2, Lemma 7.8] gives that $B(\beta * \alpha, \mathcal{W}, V)$ is mapped homeomorphically onto V by p_H. It suffices now to check that $p_H^{-1}(V)$ is the disjoint union $\coprod_{[\beta]H} B(\beta * \alpha, \mathcal{W}, V)$ indexed by cosets. The inclusion $\bigcup_\beta B(\beta * \alpha, \mathcal{W}, V) \subseteq p_H^{-1}(V)$ is obvious. If $\gamma \in P(X, x_0)$ such that $p_H([\gamma]_H) = \gamma(1) \in V$, then there is a path $\zeta \in \mathcal{U}$ such that $\zeta(1) = \gamma(1)$. Let $\beta = \gamma * \zeta^{-1}$ so that $[\gamma]_H = [\beta * \zeta]_H$ and construct $\mathcal{V}$ and $\mathcal{W}$ as above. Since $\beta * \zeta \in \mathcal{W}$, it follows that $[\gamma]_H \in B(\beta * \alpha, \mathcal{W}, V)$. Now let $[\gamma]_H \in B(\beta_1 * \alpha, \mathcal{W}_1, V) \cap B(\beta_2 * \alpha, \mathcal{W}_2, V)$ for loops $\beta_i \in \Omega(X, x_0)$ and $[\zeta]_H \in B(\beta_1 * \alpha, \mathcal{W}_1, V)$. We have $[b_1 * a_1]_H = [\gamma]_H = [b_2 * a_2]_H$ and $[\zeta]_H = [c_1 * d_1]_H$ for loops $b_1, c_1 \in \mathcal{V}_1$, $b_2 \in \mathcal{V}_2$, and paths $a_1, a_2, d_1 \in \mathcal{U}$. Since $[b_1], [c_1] \in [\beta_1]H$ and $[b_2] \in [\beta_2]H$, $[c_1 * b_1^{-1}] \in H$. Additionally, since $a_1(1) = a_2(1)$ and $a_1 * a_2^{-1} \in \mathcal{U}^{[0,\frac{1}{2}]} \cap \left(\mathcal{U}^{-1}\right)^{[\frac{1}{2},1]}$, $[a_1 * a_2^{-1}] \in H$. Note that $[b_1 * a_1 * a_2^{-1} * b_2^{-1}] \in H \cap [b_1]H[b_2^{-1}] = H \cap H[b_1][b_2^{-1}]$ and thus $[b_1 * b_2^{-1}] \in H$. Now

$$[c_1 * d_1 * d_1^{-1} * b_2^{-1}] = [c_1 * b_1^{-1}][b_1 * b_2^{-1}] \in H$$

indicates that $[\zeta]_H = [c_1 * d_1]_H = [b_2 * d_1]_H$ for $b_2 * d_1 \in \mathcal{W}_2$. Thus $[\zeta]_H \in B(\beta_2 * \alpha, \mathcal{W}_2, V)$. Since $B(\beta_1 * \alpha, \mathcal{W}_1, V) \subseteq B(\beta_2 * \alpha, \mathcal{W}_2, V)$ and p_H maps both sets homeomorphically onto V, we have $B(\beta_1 * \alpha, \mathcal{W}_1, V) = B(\beta_2 * \alpha, \mathcal{W}_2, V)$. Consequently, $p_H^{-1}(V)$ is a disjoint union and p_H is a covering map. □

2 Spanier Groups

Let X be a path connected, locally path connected space and $x_0 \in X$. In this case, Theorem 1 above is comparable with the approach of "Spanier groups." Spanier groups, first named in [3], appear in Spanier's celebrated textbook [4].

Definition 3. Let $\mathscr{U}$ be an open cover of X. The *Spanier group of* (X, x_0) *with respect to* $\mathscr{U}$ is the subgroup, denoted $\pi(\mathscr{U}, x_0)$, of $\pi_1(X, x_0)$ generated by classes of the form $[\alpha * \gamma * \alpha^{-1}]$ where $\alpha : I \to X$ is a path with $\alpha(0) = x_0$ and $\gamma \in \Omega(U, \alpha(1))$ for $U \in \mathscr{U}$.

It is evident that each group $\pi(\mathscr{U}, x_0)$ is a normal subgroup of $\pi_1(X, x_0)$. Note also that $\pi(\mathscr{U}, x_0) \subseteq \pi(\mathscr{V}, x_0)$ whenever $\mathscr{U}$ refines $\mathscr{V}$.

Proposition 4. *For every open cover $\mathscr{U}$ of X, $\pi(\mathscr{U}, x_0)$ is open in $\pi_1^\tau(X, x_0)$.*

Proof. Since X is locally path connected, there is a covering map $p : \tilde{X} \to X$ such that $p(\tilde{x}_0) = x_0$ and the image of $p_* : \pi_1^\tau(\tilde{X}, \tilde{x}_0) \to \pi_1^\tau(X, x_0)$ is $\pi(\mathscr{U}, x_0)$: See [4]. The result follows from [2] which implies that p_* is an open embedding of topological groups. □

Definition 5. If X is a path connected space and $x_0 \in X$, the *Spanier group* of X is the intersection

$$\varprojlim \pi(\mathscr{U}, x_0) = \bigcap_{\mathscr{U}} \pi(\mathscr{U}, x_0)$$

.

Theorem 6. *The Spanier group $\varprojlim \pi(\mathscr{U}, x_0)$ of X is a closed normal subgroup of $\pi_1^\tau(X, x_0)$.*

Proof. Open subgroups of topological groups are also closed. Thus by Prop. 4, $\pi(\mathscr{U}, x_0)$ is closed and normal for every open cover $\mathscr{U}$. Therefore, the intersection $\varprojlim \pi(\mathscr{U}, x_0)$ of all such groups is closed and normal. $\square$

Let $\overline{[c_{x_0}]}$ be the closure of the trivial subgroup in $\pi_1^{\tau}(X, x_0)$. An elementary property of topological groups is that the closure of the trivial subgroup in a topological group G is a normal subgroup of G.

Corollary 7. *For a locally path connected space X, $\overline{[c_{x_0}]} \subseteq \varprojlim \pi(\mathscr{U}, x_0)$.*

Corollary 8. *If $\varprojlim \pi(\mathscr{U}, x_0) = 1$, then $\pi_1^{\tau}(X, x_0)$ is a totally disconnected, Hausdorff topological group.*

Proof. By the previous corollary, the identity is closed in $\pi_1^{\tau}(X, x_0)$ and therefore $\pi_1^{\tau}(X, x_0)$ is Hausdorff. Since each open subgroup $\pi(\mathscr{U}, x_0)$ is also closed, the connected component of the identity lies in $\pi(\mathscr{U}, x_0)$ for every open cover $\mathscr{U}$. Since $\varprojlim \pi(\mathscr{U}, x_0) = 1$ the component of the identity contains only the identity. $\square$

References

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