# Open subgroups of free topological groups

Jeremy Brazas

Spring Topology and Dynamics Conference, New Britain, CT

March 23, 2013

# Free topological groups

The **free Graev topological group** on a based space $(X, e)$ is the topological group $F_G(X, e)$ equipped with a based map $\sigma : X \to F_G(X, e)$: Given $f : X \to G$, $f(e) = e_G$

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "f"'] & F_G(X,e) \arrow[d, dashed, "\exists! \widetilde{f}"] \\ & G \end{tikzcd}$$

$$\mathbf{TopGrp}(F_G(X, e), G) \cong \mathbf{Top}_*(X, U(G))$$

The unbased version is the **free Markov topological group** $F_M(X)$.

# Free topological groups

The **free Graev topological group** on a based space $(X, e)$ is the topological group $F_G(X, e)$ equipped with a based map $\sigma : X \to F_G(X, e)$:
Given $f : X \to G$, $f(e) = e_G$

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "f"'] & F_G(X,e) \arrow[d, dashed, "\exists! \widetilde{f}"] \\ & G \end{tikzcd}$$

$$\mathbf{TopGrp}(F_G(X, e), G) \cong \mathbf{Top}_*(X, U(G))$$

The unbased version is the **free Markov topological group** $F_M(X)$.

# Free topological groups

The **free Graev topological group** on a based space $(X, e)$ is the topological group $F_G(X, e)$ equipped with a based map $\sigma : X \to F_G(X, e)$:
Given $f : X \to G$, $f(e) = e_G$

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "f"'] & F_G(X,e) \arrow[d, dashed, "\exists ! \widetilde{f}"] \\ & G \end{tikzcd}$$

$$\mathbf{TopGrp}(F_G(X, e), G) \cong \mathbf{Top}_*(X, U(G))$$

The unbased version is the **free Markov topological group** $F_M(X)$.

# Free topological groups

The **free Graev topological group** on a based space $(X, e)$ is the topological group $F_G(X, e)$ equipped with a based map $\sigma : X \to F_G(X, e)$:
Given $f : X \to G$, $f(e) = e_G$

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "f"'] & F_G(X,e) \arrow[d, dashed, "\exists! \widetilde{f}"] \\ & G \end{tikzcd}$$

$$\textbf{TopGrp}(F_G(X, e), G) \cong \textbf{Top}_*(X, U(G))$$

The unbased version is the **free Markov topological group** $F_M(X)$.

# Free topological groups

The **free Graev topological group** on a based space $(X, e)$ is the topological group $F_G(X, e)$ equipped with a based map $\sigma : X \to F_G(X, e)$:
Given $f : X \to G$, $f(e) = e_G$

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "f"'] & F_G(X,e) \arrow[d, dashed, "\exists! \widetilde{f}"] \\ & G \end{tikzcd}$$

$$\mathbf{TopGrp}(F_G(X, e), G) \cong \mathbf{Top}_*(X, U(G))$$

The unbased version is the **free Markov topological group** $F_M(X)$.

# Free topological groups

- Introduced by

  Markov (1940s) $F_M(X)$
  Graev (1960s) $F_G(X, e)$

# Free topological groups

- Introduced by

  Markov (1940s) $F_M(X)$
  Graev (1960s) $F_G(X, e)$

- Free topological groups take the place of free groups in the general study of topological groups.

# Free topological groups

- Introduced by

  Markov (1940s) $F_M(X)$
  Graev (1960s) $F_G(X, e)$

- Free topological groups take the place of free groups in the general study of topological groups.
- Typically, $X$ is Tychonoff ($\sigma : X \subset F_G(X, e)$).

# Free topological groups

- Introduced by

  Markov (1940s) $F_M(X)$
  Graev (1960s) $F_G(X,e)$

- Free topological groups take the place of free groups in the general study of topological groups.
- Typically, $X$ is Tychonoff ($\sigma : X \subset F_G(X,e)$).
- As groups,

$$F_M(X) = F(X)$$
$$F_G(X,e) = F(X \backslash e) = F(X)/\langle e \rangle^{F(x)}$$

# Topological Nielsen-Schreier?

**Nielsen-Schreier Theorem:** Every subgroup of a free group is free.

# Topological Nielsen-Schreier?

**Nielsen-Schreier Theorem:** Every subgroup of a free group is free.

Question: Is every subgroup $H \leq F_G(X, e)$ a free Graev topological group?

# Topological Nielsen-Schreier?

**Nielsen-Schreier Theorem:** Every subgroup of a free group is free.

Question: Is every subgroup $H \leq F_G(X, e)$ a free Graev topological group?

No, not even if $H$ is closed (Graev; Brown; Clarke; Hunt, Morris).

# Topological Nielsen-Schreier?

**Nielsen-Schreier Theorem:** Every subgroup of a free group is free.

Question: Is every subgroup $H \leq F_G(X, e)$ a free Graev topological group?

No, not even if $H$ is closed (Graev; Brown; Clarke; Hunt, Morris).

Yes, if $X$ is *Hausdorff* $k_\omega$*-space* and $H$ is **open** (R. Brown, J.P. Hardy; P. Nickolas).

# Topological Nielsen-Schreier?

**Nielsen-Schreier Theorem:** Every subgroup of a free group is free.

Question: Is every subgroup $H \leq F_G(X, e)$ a free Graev topological group?

No, not even if $H$ is closed (Graev; Brown; Clarke; Hunt, Morris).

Yes, if $X$ is *Hausdorff $k_\omega$-space* and $H$ is **open** (R. Brown, J.P. Hardy; P. Nickolas).

Yes, in abelian case $A_G(X, e)$, $X$ Tychonoff and $H$ **open** (Morris, Pestov).

# Topological Nielsen-Schreier?

**Nielsen-Schreier Theorem:** Every subgroup of a free group is free.

Question: Is every subgroup $H \leq F_G(X,e)$ a free Graev topological group?

No, not even if $H$ is closed (Graev; Brown; Clarke; Hunt, Morris).

Yes, if $X$ is *Hausdorff $k_\omega$-space* and $H$ is **open** (R. Brown, J.P. Hardy; P. Nickolas).

Yes, in abelian case $A_G(X,e)$, $X$ Tychonoff and $H$ **open** (Morris, Pestov).

Question: Is every open subgroup $H \leq F_G(X,e)$ a free Graev topological group? (at least for Tychonoff $X$)

# Main Results

**Theorem 1:** Every open subgroup of a free Graev topological group is a free Graev topological group.

**Theorem 2:** Every open subgroup of a free Markov topological group is a free Markov topological group iff it is disconnected.

*No separation axioms are required.

# Main Results

**Theorem 1:** Every open subgroup of a free Graev topological group is a free Graev topological group.

**Theorem 2:** Every open subgroup of a free Markov topological group is a free Markov topological group iff it is disconnected.

*No separation axioms are required.

# Putting algebraic topology to work

**Algebraic topology:** The *fundamental group* $\pi_1$ and *covering space theory* provide a straightforward proof of the Nielsen-Schreier Theorem.

$$\text{Topology} \longleftrightarrow \text{Algebra}$$

**Extension:** Use a topologically enriched version $\pi_1^\tau$ of the fundamental group and a generalization of covering spaces (semicovering spaces).

$$\text{“Wild” Topology} \longleftrightarrow \text{Topological algebra}$$

# Putting algebraic topology to work

**Algebraic topology:** The *fundamental group* $\pi_1$ and *covering space theory* provide a straightforward proof of the Nielsen-Schreier Theorem.

$$\text{Topology} \longleftrightarrow \text{Algebra}$$

**Extension:** Use a topologically enriched version $\pi_1^\tau$ of the fundamental group and a generalization of covering spaces (semicovering spaces).

$$\text{“Wild” Topology} \longleftrightarrow \text{Topological algebra}$$

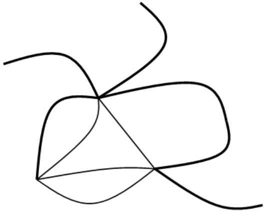
# The Nielsen-Schreier Theorem

**Nielsen-Schreier Theorem:** Every subgroup of a free group is free.

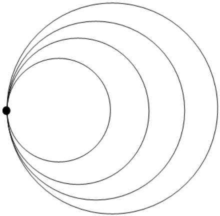
# The Nielsen-Schreier Theorem

**Nielsen-Schreier Theorem:** Every subgroup of a free group is free.

**Lemma:** The fundamental group of a graph is free.



$\simeq$



# Completing the proof.

*Proof.*

Suppose $H \leq F(A) = \pi_1\left(\bigvee_A S^1, x_0\right)$

There is a covering map $p : Y \to \bigvee_A S^1$ such that $p_*(\pi_1(Y, y_0)) = H$.

The covering of a graph is a graph, so $Y$ is a graph.

Since $Y$ is a graph, $H \cong \pi_1(Y, y_0)$ is free.

# Completing the proof.

*Proof.*

Suppose $H \leq F(A) = \pi_1\left(\bigvee_A S^1, x_0\right)$

There is a covering map $p : Y \to \bigvee_A S^1$ such that $p_*(\pi_1(Y, y_0)) = H$.

The covering of a graph is a graph, so $Y$ is a graph.

Since $Y$ is a graph, $H \cong \pi_1(Y, y_0)$ is free.

# Completing the proof.

*Proof.*

Suppose $H \leq F(A) = \pi_1\left(\bigvee_A S^1, x_0\right)$

There is a covering map $p : Y \to \bigvee_A S^1$ such that $p_*(\pi_1(Y, y_0)) = H$.

The covering of a graph is a graph, so $Y$ is a graph.

Since $Y$ is a graph, $H \cong \pi_1(Y, y_0)$ is free.

# Completing the proof.

*Proof.*

Suppose $H \leq F(A) = \pi_1\left(\bigvee_A S^1, x_0\right)$

There is a covering map $p : Y \to \bigvee_A S^1$ such that $p_*(\pi_1(Y, y_0)) = H$.

The covering of a graph is a graph, so $Y$ is a graph.

Since $Y$ is a graph, $H \cong \pi_1(Y, y_0)$ is free.

# Completing the proof.

*Proof.*

Suppose $H \leq F(A) = \pi_1\left(\bigvee_A S^1, x_0\right)$

There is a covering map $p : Y \to \bigvee_A S^1$ such that $p_*(\pi_1(Y, y_0)) = H$.

The covering of a graph is a graph, so $Y$ is a graph.

Since $Y$ is a graph, $H \cong \pi_1(Y, y_0)$ is free.

# Completing the proof.

*Proof.*

Suppose $H \leq F(A) = \pi_1\left(\bigvee_A S^1, x_0\right)$

There is a covering map $p : Y \to \bigvee_A S^1$ such that $p_*(\pi_1(Y, y_0)) = H$.

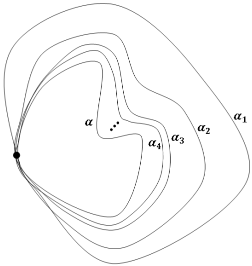
The covering of a graph is a graph, so $Y$ is a graph.

Since $Y$ is a graph, $H \cong \pi_1(Y, y_0)$ is free.

# Topologizing $\pi_1$

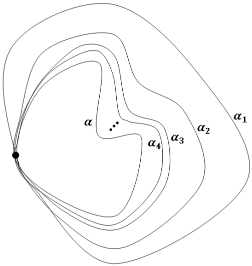
# Topologizing $\pi_1$

**Guiding principle:** If $\alpha_n \to \alpha$ in $\Omega(X, x)$, then $[\alpha_n] \to [\alpha]$ in $\pi_1(X, x)$.



# Topologizing $\pi_1$

**Guiding principle:** If $\alpha_n \to \alpha$ in $\Omega(X,x)$, then $[\alpha_n] \to [\alpha]$ in $\pi_1(X,x)$.




$\Omega(X,x) \to \pi_1(X,x)$, $\alpha \mapsto [\alpha]$ should be continuous.

# The quasitopological fundamental group

**Natural choice:** Give $\pi_1(X,x)$ the quotient topology with respect to $\Omega(X,x) \to \pi_1(X,x)$ (Biss).

- $\pi_1^{qtop}(X,x)$ is a quasitopological group (Calcut, McCarthy).
- $\pi_1^{qtop}(X,x)$ can fail to be a topological group even for Peano continua (P. Fabel).
- The topology of $\pi_1^{qtop}(X,x)$ is often complicated but can retain more local data than shape invariants.

A left adjoint $\tau$ : **qTopGrp** $\to$ **TopGrp** removes the "fewest number" of open sets from quasitopological $G$ until a topological $\tau(G)$ is obtained.

$$\text{Let } \pi_1^{\tau}(X,x) = \tau(\pi_1^{qtop}(X,x))$$

# The quasitopological fundamental group

**Natural choice:** Give $\pi_1(X,x)$ the quotient topology with respect to $\Omega(X,x) \to \pi_1(X,x)$ (Biss).

- $\pi_1^{qtop}(X,x)$ is a quasitopological group (Calcut, McCarthy).
- $\pi_1^{qtop}(X,x)$ can fail to be a topological group even for Peano continua (P. Fabel).
- The topology of $\pi_1^{qtop}(X,x)$ is often complicated but can retain more local data than shape invariants.

A left adjoint $\tau$ : **qTopGrp** $\to$ **TopGrp** removes the "fewest number" of open sets from quasitopological $G$ until a topological $\tau(G)$ is obtained.

$$\text{Let } \pi_1^{\tau}(X,x) = \tau(\pi_1^{qtop}(X,x))$$

# The quasitopological fundamental group

**Natural choice:** Give $\pi_1(X,x)$ the quotient topology with respect to $\Omega(X,x) \to \pi_1(X,x)$ (Biss).

- $\pi_1^{qtop}(X,x)$ is a quasitopological group (Calcut, McCarthy).
- $\pi_1^{qtop}(X,x)$ can fail to be a topological group even for Peano continua (P. Fabel).
- The topology of $\pi_1^{qtop}(X,x)$ is often complicated but can retain more local data than shape invariants.

A left adjoint $\tau$ : **qTopGrp** $\to$ **TopGrp** removes the "fewest number" of open sets from quasitopological $G$ until a topological $\tau(G)$ is obtained.

$$\text{Let } \pi_1^{\tau}(X,x) = \tau(\pi_1^{qtop}(X,x))$$

# The quasitopological fundamental group

**Natural choice:** Give $\pi_1(X,x)$ the quotient topology with respect to $\Omega(X,x) \to \pi_1(X,x)$ (Biss).

- $\pi_1^{qtop}(X,x)$ is a quasitopological group (Calcut, McCarthy).
- $\pi_1^{qtop}(X,x)$ can fail to be a topological group even for Peano continua (P. Fabel).
- The topology of $\pi_1^{qtop}(X,x)$ is often complicated but can retain more local data than shape invariants.

A left adjoint $\tau$ : **qTopGrp** $\to$ **TopGrp** removes the "fewest number" of open sets from quasitopological $G$ until a topological $\tau(G)$ is obtained.

Let $\pi_1^{\tau}(X,x) = \tau(\pi_1^{qtop}(X,x))$

# The quasitopological fundamental group

**Natural choice:** Give $\pi_1(X,x)$ the quotient topology with respect to $\Omega(X,x) \to \pi_1(X,x)$ (Biss).

- $\pi_1^{qtop}(X,x)$ is a quasitopological group (Calcut, McCarthy).
- $\pi_1^{qtop}(X,x)$ can fail to be a topological group even for Peano continua (P. Fabel).
- The topology of $\pi_1^{qtop}(X,x)$ is often complicated but can retain more local data than shape invariants.

A left adjoint $\tau$ : **qTopGrp** $\to$ **TopGrp** removes the "fewest number" of open sets from quasitopological $G$ until a topological $\tau(G)$ is obtained.

$$\text{Let } \pi_1^{\tau}(X,x) = \tau(\pi_1^{qtop}(X,x))$$

# Topological $\pi_1$

**Characterizations:**

- The topology of $\pi_1^{\tau}(X,x)$ is the finest *group* topology on $\pi_1(X,x)$ such that $\Omega(X,x) \to \pi_1(X,x)$ is continuous.

# Topological $\pi_1$

**Characterizations:**

- The topology of $\pi_1^\tau(X,x)$ is the finest *group* topology on $\pi_1(X,x)$ such that $\Omega(X,x) \to \pi_1(X,x)$ is continuous.
- Transfinite approximation by quasitopological groups

# Topological $\pi_1$

**Characterizations:**

- The topology of $\pi_1^\tau(X,x)$ is the finest *group* topology on $\pi_1(X,x)$ such that $\Omega(X,x) \to \pi_1(X,x)$ is continuous.
- Transfinite approximation by quasitopological groups

**Utility: realizing universal topological groups**

- $\pi_1^\tau(\Sigma(Z_+)) \cong F_M(\pi_0^{qtop}(Z))$ where $\Sigma(Z_+) = Z \times I / Z \times \{0,1\}$ is a generalized wedge of circles.
- Topological van-Kampen theorems

# Semicovering maps

**Definition:** A map $p : Y \to X$ is a semicovering map if

- $p$ is a local homeomorphism
- Whenever $f$ is a path or homotopy of paths starting at $p(y_0) = x_0$, there is a *unique* lift $\widetilde{f}$ starting at $y_0$
- Each lifting function $f \mapsto \widetilde{f}$ is continuous on mapping spaces.

Every covering map is a semicovering map but not every semicovering is a covering map.

# Semicovering maps

**Definition:** A map $p : Y \to X$ is a semicovering map if

- $p$ is a local homeomorphism
- Whenever $f$ is a path or homotopy of paths starting at $p(y_0) = x_0$, there is a *unique* lift $\widetilde{f}$ starting at $y_0$
- Each lifting function $f \mapsto \widetilde{f}$ is continuous on mapping spaces.

Every covering map is a semicovering map but not every semicovering is a covering map.

# Semicovering maps

**Definition:** A map $p : Y \to X$ is a semicovering map if

- $p$ is a local homeomorphism
- Whenever $f$ is a path or homotopy of paths starting at $p(y_0) = x_0$, there is a *unique* lift $\widetilde{f}$ starting at $y_0$
- Each lifting function $f \mapsto \widetilde{f}$ is continuous on mapping spaces.

Every covering map is a semicovering map but not every semicovering is a covering map.

# Semicovering maps

**Definition:** A map $p : Y \to X$ is a semicovering map if

- $p$ is a local homeomorphism
- Whenever $f$ is a path or homotopy of paths starting at $p(y_0) = x_0$, there is a *unique* lift $\widetilde{f}$ starting at $y_0$
- Each lifting function $f \mapsto \widetilde{f}$ is continuous on mapping spaces.

Every covering map is a semicovering map but not every semicovering is a covering map.

# Semicovering maps

**Definition:** A map $p : Y \to X$ is a semicovering map if

- $p$ is a local homeomorphism
- Whenever $f$ is a path or homotopy of paths starting at $p(y_0) = x_0$, there is a *unique* lift $\widetilde{f}$ starting at $y_0$
- Each lifting function $f \mapsto \widetilde{f}$ is continuous on mapping spaces.

Every covering map is a semicovering map but not every semicovering is a covering map.

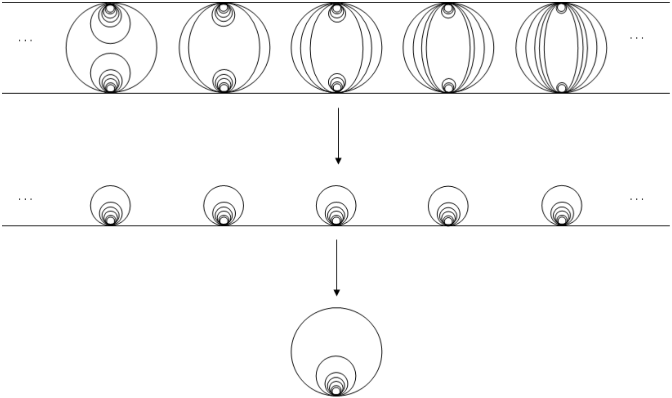
# Semicovering maps

**Definition:** A map $p : Y \to X$ is a semicovering map if

- $p$ is a local homeomorphism
- Whenever $f$ is a path or homotopy of paths starting at $p(y_0) = x_0$, there is a *unique* lift $\widetilde{f}$ starting at $y_0$
- Each lifting function $f \mapsto \widetilde{f}$ is continuous on mapping spaces.

Every covering map is a semicovering map but not every semicovering is a covering map.

# A semicovering which is not a covering



# Traditional Covering Theory

A covering map $p : Y \to X$ induces an injection $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ of groups. Let $H = p_*(\pi_1(Y, y_0))$.

**Classification of covering maps:** If $X$ is locally "nice" (locally path connected, and semilocally simply connected), then there is a bijective correspondence

$$\left\{\begin{matrix}\text{Equivalence classes of}\\ \text{coverings } p : Y \to X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Conjugacy classes of}\\ \text{subgroups } H \leq \pi_1(X, x)\end{matrix}\right\}$$

# Semicovering Space Theory

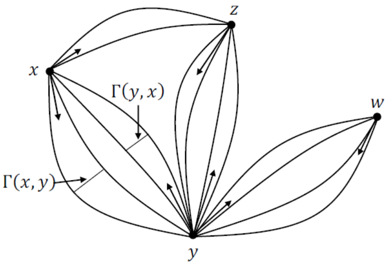
A semicovering map $p : Y \to X$ induces an **open embedding** $p_* : \pi_1^\tau(Y, y_0) \to \pi_1^\tau(X, x_0)$ of topological groups. Let $H = p_*(\pi_1^\tau(Y, y_0))$.

**Classification of semicovering maps:** If $X$ is locally wep-connected, then there is a bijective correspondence

$$\left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{semicoverings } p : Y \to X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Conjugacy classes of} \\ \textbf{open} \text{ subgroups } H \leq \pi_1^\tau(X, x) \end{array} \right\}$$

# **Top**-graphs

Replacement of graphs: A **Top**-graph $\Gamma$ consists of a discrete space of vertices and an edge space $\Gamma(x, y)$ for each ordered pair of vertices $(x, y)$.

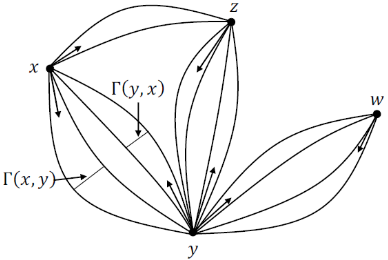



**Lemma:** If $\Gamma$ is a **Top**-graph, then $\pi_1^\tau(\Gamma, x)$ is a free Graev topological group.

*Proof.* Use universal constructions of topologically enriched categories and groupoids (i.e. **Top**-categories and **Top**-groupoids).

# **Top**-graphs

Replacement of graphs: A **Top**-graph $\Gamma$ consists of a discrete space of vertices and an edge space $\Gamma(x,y)$ for each ordered pair of vertices $(x,y)$.

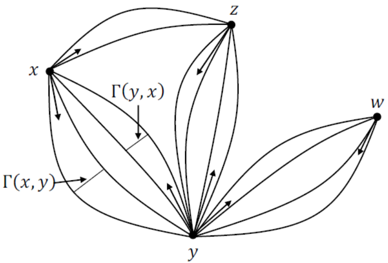



**Lemma:** If $\Gamma$ is a **Top**-graph, then $\pi_1^\tau(\Gamma, x)$ is a free Graev topological group.

*Proof.* Use universal constructions of topologically enriched categories and groupoids (i.e. **Top**-categories and **Top**-groupoids).

# **Top**-graphs

Replacement of graphs: A **Top**-graph $\Gamma$ consists of a discrete space of vertices and an edge space $\Gamma(x, y)$ for each ordered pair of vertices $(x, y)$.




**Lemma:** If $\Gamma$ is a **Top**-graph, then $\pi_1^\tau(\Gamma, x)$ is a free Graev topological group.

*Proof.* Use universal constructions of topologically enriched categories and groupoids (i.e. **Top**-categories and **Top**-groupoids).

# Proof sketch

Suppose $H \leq F_G(X, e)$ is open.

- Find $Z$ such that $\pi_0^{qtop}(Z) \cong X$ (D. Harris)
- The unreduced suspension $\Gamma = SZ$ is a **Top**-graph such that $\pi_1^\tau(\Gamma, x) = F_G(X, e)$.
- There is a semicovering $p : Y \to \Gamma$ such that $p_*(\pi_1^\tau(Y, y)) = H$.
- A semicovering of a **Top**-graph is a **Top**-graph, so $Y$ is a **Top**-graph.
- Since $Y$ is a **Top**-graph, $p_* : \pi_1^\tau(Y, y) \cong H$ is free Graev topological.

# Proof sketch

Suppose $H \leq F_G(X, e)$ is open.

Find $Z$ such that $\pi_0^{qtop}(Z) \cong X$ (D. Harris)

The unreduced suspension $\Gamma = SZ$ is a **Top**-graph such that $\pi_1^\tau(\Gamma, x) = F_G(X, e)$.

There is a semicovering $p : Y \to \Gamma$ such that $p_*(\pi_1^\tau(Y, y)) = H$.

A semicovering of a **Top**-graph is a **Top**-graph, so $Y$ is a **Top**-graph.

Since $Y$ is a **Top**-graph, $p_* : \pi_1^\tau(Y, y) \cong H$ is free Graev topological.

# Proof sketch

Suppose $H \leq F_G(X, e)$ is open.

- Find $Z$ such that $\pi_0^{qtop}(Z) \cong X$ (D. Harris)
- The unreduced suspension $\Gamma = SZ$ is a **Top**-graph such that $\pi_1^{\tau}(\Gamma, x) = F_G(X, e)$.
- There is a semicovering $p : Y \to \Gamma$ such that $p_*(\pi_1^{\tau}(Y, y)) = H$.
- A semicovering of a **Top**-graph is a **Top**-graph, so $Y$ is a **Top**-graph.
- Since $Y$ is a **Top**-graph, $p_* : \pi_1^{\tau}(Y, y) \cong H$ is free Graev topological.

# Proof sketch

Suppose $H \leq F_G(X, e)$ is open.

- Find $Z$ such that $\pi_0^{qtop}(Z) \cong X$ (D. Harris)
- The unreduced suspension $\Gamma = SZ$ is a **Top**-graph such that $\pi_1^\tau(\Gamma, x) = F_G(X, e)$.
- There is a semicovering $p : Y \to \Gamma$ such that $p_*(\pi_1^\tau(Y, y)) = H$.
- A semicovering of a **Top**-graph is a **Top**-graph, so $Y$ is a **Top**-graph.
- Since $Y$ is a **Top**-graph, $p_* : \pi_1^\tau(Y, y) \cong H$ is free Graev topological.

# Proof sketch

Suppose $H \leq F_G(X,e)$ is open.

- Find $Z$ such that $\pi_0^{qtop}(Z) \cong X$ (D. Harris)
- The unreduced suspension $\Gamma = SZ$ is a **Top**-graph such that $\pi_1^{\tau}(\Gamma,x) = F_G(X,e)$.
- There is a semicovering $p : Y \to \Gamma$ such that $p_*(\pi_1^{\tau}(Y,y)) = H$.
- A semicovering of a **Top**-graph is a **Top**-graph, so $Y$ is a **Top**-graph.
- Since $Y$ is a **Top**-graph, $p_* : \pi_1^{\tau}(Y,y) \cong H$ is free Graev topological.

# Proof sketch

Suppose $H \leq F_G(X, e)$ is open.

- Find $Z$ such that $\pi_0^{qtop}(Z) \cong X$ (D. Harris)
- The unreduced suspension $\Gamma = SZ$ is a **Top**-graph such that $\pi_1^\tau(\Gamma, x) = F_G(X, e)$.
- There is a semicovering $p : Y \to \Gamma$ such that $p_*(\pi_1^\tau(Y, y)) = H$.
- A semicovering of a **Top**-graph is a **Top**-graph, so $Y$ is a **Top**-graph.
- Since $Y$ is a **Top**-graph, $p_* : \pi_1^\tau(Y, y) \cong H$ is free Graev topological.

# References

J. Brazas, *Semicoverings: A generalization of covering space theory*. Homology Homotopy Appl. 14 (2012) 33-63.

J. Brazas, *The fundamental group as topological group*. Topology Appl. 160 (2013) 170-188.

J. Brazas, *Open subgroups of free topological groups*. Submitted. 2013. arXiv:1209.5486.

R. Brown, J.P.L. Hardy, *Subgroups of free topological groups and free topological products of topological groups*, J. London Math. Soc. (2) 10 (1975) 431-440.

P. Fabel, *Multiplication is discontinuous in the Hawaiian earring group*. Bull. Polish Acad. Sci. Math. 59 (2011) 7783.

Graev, M.I. *Free topological groups*. Amer. Math. Soc. Transl. 8 (1962) 305-365.

Markov, A.A. *On free topological groups*. Izv. Akad. Nauk. SSSR Ser. Mat. 9 (1945) 3-64.

S.A. Morris, V.G. Pestov, *Open subgroups of free abelian topological groups*. Math. Proc. Camb. Phil. Soc. 114 (1993) 439-442.

P. Nickolas, *A Schreier theorem for free topological groups*, Bulletin of the Australian Math. Soc. 13 (1975) 121–127.

O. Schreier, *Die Untergruppen der freien Gruppen*, Abh. Mat Sem. Univ. Hamburg 3 (1927) 167-169.