THE LOCALLY PATH-CONNECTED COREFLECTION

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Abstract. Given any topological space X, it is possible to construct a locally path connected space $\mathrm{lpc}(X)$ with the underlying set of X so that the identity $id : \mathrm{lpc}(X) \to X$ is continuous and universal with respect to maps $f : Z \to X$ from all locally path connected spaces. This construction defines a functor $\mathrm{lpc} : \mathbf{Top} \to \mathbf{LC}_0$ from the topological category to the subcategory of locally path connected spaces, which is right adjoint to the inclusion $\mathbf{LC}_0 \to \mathbf{Top}$. This expository note includes the basic topological properties of this construction and some of its applications in algebraic topology.

1. Introduction

The construction sharing its name with the title of this note has been around for a long time and the author claims no originality in the content of this paper. Coreflection functors are quite common to use in topology, e.g. for k-spaces [4], locally connected spaces [7], etc. Coreflection constructions are highly useful because they take a space X, which may not be an object in a desired subcategory $\mathscr{C}$ of **Top** and constructs an object $c(X) \in \mathscr{C}$ in the "most efficient way possible" simply by refining the topology of X. This becomes particularly helpful when say a product $X \times Y$ of objects $X, Y \in \mathscr{C}$ with the product topology is not actually an object of $\mathscr{C}$. Thus $c(X \times Y)$ becomes the categorical product in $\mathscr{C}$. When one learns that the product of two locally path connected spaces (with the product topology) is still locally path connected, it seems possible that the locally path connected category $\mathbf{LC}_0$ is better behaved. However, in the end $\mathbf{LC}_0$ is not closed under infinite products (unless the spaces themselves are also path connected), equalizers, and pretty much any other types of limit.

All category theory books discuss adjoint functors, e.g. [8], and some give reflections and coreflections special attention, e.g. [1]. Because of the prominence of limits of Peano continua or of covering spaces in the algebraic topology of locally complicated spaces, the locally path-connected coreflection has become a fairly important tool. This note is intended to be an elementary and detailed reference on the properties of the locally path-connected coreflection construction, which would likely be too specific for a category theory text to include.

2. Constructing $\mathrm{lpc}(X)$

It is well-known that a topological space X is locally path-connected if and only if every path component of every open set in X is open. We allow this fact to

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motivate our construction of a locally-path connected space $\text{lpc}(X)$ even if X is not locally path connected.

Given a topological space X, let $\mathscr{B}(X)$ be the set of path components of open sets in X.

Proposition 2.1. *$\mathscr{B}(X)$ is a basis for a topology on the underlying set of X.*

Proof. Certainly every point of x is contained is some path component of X. Suppose U_1 and U_2 are open in X and $x \in U_1 \cap U_2$. Let C_i be the path component of x in U_i for $i \in \{1, 2\}$ and C be the path component of x in the intersection $U_1 \cap U_2$. It suffices to show that $C \subseteq C_1 \cap C_2$. If $c \in C$, then there is a path γ from c to x in $U_1 \cap U_2$. Since C_i is the path component of x in U_i, the path γ must have image in both C_1 and C_2. Thus $c \in C_1 \cap C_2$. □

Definition 2.2. Given a topological space X, let $\text{lpc}(X)$ be the space with the same underlying set as X and with the topology generated by $\mathscr{B}(X)$. We refer to $\text{lpc}(X)$ as the *locally path-connected coreflection* of X or simply as the *lpc-coreflection* of X.

In [3], $\text{lpc}(X)$ is referred to as the *universal* lpc-*space*. When X is also path connected, these authors refer to $\text{lpc}(X)$ as the *Peanification* of X. See also [6].

Remark 2.3. Since every open set in X is the union of its path components, the topology of $\text{lpc}(X)$ is finer than the topology of X. Equivalently, the identity function $id : \text{lpc}(X) \to X$ is continuous.

The following, is the most important property of $\text{lpc}(X)$.

Lemma 2.4. *Suppose X is a space, Y is a locally path connected space, and $f : Y \to X$ is a function. Then $f : Y \to X$ is continuous if and only if $f : Y \to lpc(X)$ is continuous.*

Proof. Since the topology of $\text{lpc}(X)$ is finer than that of X, one direction is clear.

Let $f : Y \to X$ be a continuous function. To show that $f : Y \to \text{lpc}(X)$ is continuous, suppose C is the path component of an open subset U of X (so that C is a basic open set in $\text{lpc}(X)$). Suppose $y \in Y$ such that $f(y) \in C$. Since $f : Y \to X$ is continuous and U is an open neighborhood of $f(y)$ in X, there is an open neighborhood V of y in Y such that $f(V) \subseteq U$. Now since Y is locally path connected, we may find a path connected open set W in Y such that $y \in W \subseteq V$. It suffices to check that $f(W) \subseteq C$. If $w \in W$, then there is a path $\gamma : [0, 1] \to W$ from y to w. Now $f \circ \gamma : [0, 1] \to f(W) \subseteq f(V) \subseteq U$ is a path from $f(y)$ to $f(w)$. Since C is the path component of $f(y)$ in U, we must have $f(w) \in C$. This proves $f(W) \subseteq C$. We conclude that $f : Y \to \text{lpc}(X)$ is continuous. □

Another way to think about this is in terms of hom-sets of continuous functions. Here **Top** denote the category of topological spaces and continuous functions and $\mathbf{LC}_0$ is the full subcategory of locally path connected spaces. Thus $\mathbf{Top}(A, B)$ denotes the set of all continuous functions $A \to B$ and $\mathbf{LC}_0(A, B) = \mathbf{Top}(A, B)$ when A and B are locally path connected.

Corollary 2.5. *If Y is locally path connected, then the continuous identity $id : \text{lpc}(X) \to X$ induces a bijection $\eta : \mathbf{Top}(Y, \text{lpc}(X)) \to \mathbf{Top}(Y, X)$ given by composing a map $Y \to \text{lpc}(X)$ with $id : \text{lpc}(X) \to X$.*

Proof. Injectivity of η follows from the injectivity of the identity function and surjectivity of η follows from Lemma 2.4. □

An important case of Lemma 2.4 is when we take $Y = [0,1]$ to be the unit interval. In this case, the above corollary can be interpreted as the fact that X and $lpc(X)$ have the same paths and homotopies of paths. For one, if X is path connected, then so is $lpc(X)$.

The original intent was to construct a locally path connected version of a space X in an "efficient" way. Let's continue to check that we've actually done this. Although it may feel obvious that $\mathrm{lpc}(X)$, this must actually be checked. Indeed, if one performs the same construction for local connectedness (using connected components of open sets), the resulting space need not be locally connected.

Proposition 2.6. *For any space X, $\mathrm{lpc}(X)$ is locally path connected. Moreover, $\mathrm{lpc}(X) = X$ if and only if X is locally path connected.*

Proof. Suppose C is a basic open neighborhood of a point $x \in \mathrm{lpc}(X)$. By construction of $\mathrm{lpc}(X)$, C is the path component of an open neighborhood in X. It is important to notice here that the subspace topologies with respect to the topologies of X and $\mathrm{lpc}(X)$ may be different so we must check that C is path connected as a subspace of $\mathrm{lpc}(X)$.

Let $x, y \in C$. Then there is a path $\gamma : [0,1] \to X$ with image in C and also $\gamma(0) = x$ and $\gamma(1) = y$. Since $\gamma : [0,1] \to \mathrm{lpc}(X)$ is also continuous (Lemma 2.4) and has image in the subset C, we can conclude that x and y can be connect by a path in the subspace C of $\mathrm{lpc}(X)$. Thus C is indeed path connected as a subspace of $\mathrm{lpc}(X)$ confirming that $\mathrm{lpc}(X)$ is indeed locally path connected.

For the second statement, it is now clear that if $\mathrm{lpc}(X) = X$, then X is locally path connected. Conversely, if X is locally path connected, then according to Lemma 2.4 that the continuity of the identity function $X \to X$ implies the continuity of the identity function $X \to \mathrm{lpc}(X)$. We already knew the identity $\mathrm{lpc}(X) \to X$ was continuous so the topologies of X and $\mathrm{lpc}(X)$ must be equal. □

Now that we know $\mathrm{lpc}(X)$ is always locally path connected, Corollary 2.5 turns into the following theorem.

Theorem 2.7. *If Y is locally path connected, then the continuous identity $id : \mathrm{lpc}(X) \to X$ induces a bijection $\eta : \mathbf{LC}_0(Y, \mathrm{lpc}(X)) \to \mathbf{Top}(Y, X)$ given by composing a map $Y \to \mathrm{lpc}(X)$ with $id : \mathrm{lpc}(X) \to X$.*

2.1. **Examples.** On one hand, if X is already locally path-connected (including any CW-complex, manifold, etc.) then $\mathrm{lpc}(X) = X$. At the other extreme, we have the following.

Example 2.8. If X is a totally path-disconnected space, such as $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$, an ordinal, the Cantor set, a pseudoarc, etc., then $\mathrm{lpc}(X)$ is a discrete space.

Remark 2.9 (lpc separates path components)**.** The path components of a locally path connected space are open. Therefore, if $\{C_j \mid j \in J\}$ are the path components of a space X, then $\mathrm{lpc}(X) = \coprod_{j \in J} \mathrm{lpc}(C_j)$ is the disjoint union of the lpc-coreflections of the path components. Intuitively, lpc separates the path components from each other.

For example, if $T = A \cup B$ is the closed topologist's since curve with $A = \{0\} \times [-1,1]$ and $B = \{(x, \sin(1/x)) \mid 0 < x \leqslant 1\}$, then $\mathrm{lpc}(T)$ is the disjoint union

$A \sqcup B$ (Since A and B are already locally path connected themselves). In particular, $\mathrm{lpc}(T)$ is the disjoint union of a closed interval and a half-closed interval.

For more enlightening examples, we now turn to path-connected, non-locally path-connected spaces.

Example 2.10 (Fan space)**.** Given $(x, y) \in \mathbb{R}^2$, let $L(x, y)$ be the closed line segment from the origin to $L(x, y)$. Take $X = L(1, 0) \cup \bigcup_{n \in \mathbb{N}} L(1, 1/n)$ and notice that X is not locally path-connected at any point $(x, 0)$, $0 < x \leqslant 1$. Applying the locally path-connected coreflection has the effect of "moving" $L(1, 0)$ away from the rest of the line segments so that the segments $L(1, 1/n)$ no longer converge.

Indeed, we can construct a homeomorphism $\mathrm{lpc}(X) \cong L(-1, -1) \cup \bigcup_{n \in \mathbb{N}} L(1, 1/n)$, which maps $L(1, 0)$ to $L(-1, 1)$ and is the identity on each $L(1, 1/n)$.

For a real number $r > 0$, let $C_r = \{(x, y) \mid (x - r)^2 + y^2 = r^2\}$ be the circle of radius r centered at $(r, 0)$. Additionally, if $A \subseteq (0, \infty)$, let $C_A = \displaystyle\bigcup_{r \in A} C_r$.

Example 2.11. Let $A = \left\{1, \ldots, 1 + \frac{1}{4}, 1 + \frac{1}{3}, 1 + \frac{1}{2}, 2\right\}$. Then $X = C_A$ is a non-locally path-connected, compact planar set (see Figure 1)

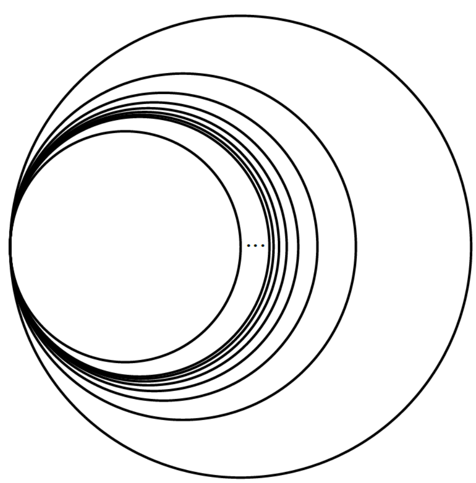

FIGURE 1.

In the space $\mathrm{lpc}(X)$, the circle $C_{1+\frac{1}{n}}$ no longer converge to C_1. Indeed, we can construct a homeomorphism $lpc(X) \cong C_B$ where $B = \left\{\frac{1}{2}, \ldots, 1 + \frac{1}{4}, 1 + \frac{1}{3}, 1 + \frac{1}{2}, 2\right\}$ is discrete (see Figure 2).

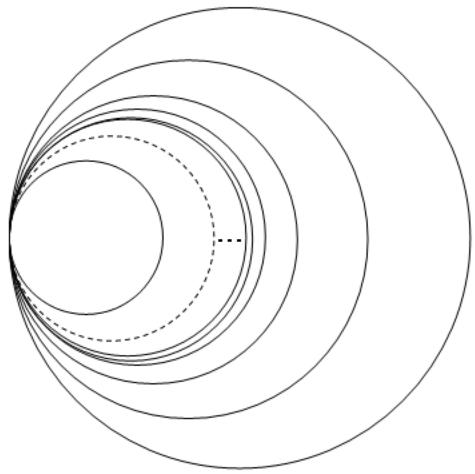

FIGURE 2. The lpc-coreflection of the space in Figure 1

The spaces C_A and C_B are not homotopy equivalent. This shows that lpc does not always preserve the homotopy type of a space.

3. A categorical viewpoint

3.1. lpc **as a coreflection functor.** The construction of $\mathrm{lpc}(X)$ is a special type of functor called a coreflection function - the idea being that the category $\mathbf{LC}_0$ of locally path-connected spaces is a subcategory of **Top** such that for every object of **Top** there is a "most efficient" way to construct a corresponding object of $\mathbf{LC}_0$.

Definition 3.1. Suppose $\mathcal{C}$ is a category and $\mathcal{D}$ is a subcategory. We say $\mathcal{D}$ is a *coreflective subcategory* of $\mathcal{C}$ if the inclusion functor $\mathcal{D} \to \mathcal{C}$ has a right adjoint $R : \mathcal{C} \to \mathcal{D}$ called a *coreflection functor*.

If we break this definition down, the fact that R is right adjoint to the inclusion means that for every object c of $\mathcal{C}$, there is an object $R(c)$ of $\mathcal{D}$ and a morphism $\eta : c \to R(c)$ in $\mathcal{C}$ which induces a bijection

$$\mathcal{C}(d, c) \to \mathcal{D}(d, R(c)) \text{ where } f \to \eta \circ f$$

for every object d of $\mathcal{D}$. This is precisely our situation: $R = lpc$ and $\eta = id : \mathrm{lpc}(X) \to X$ is the continuous identity.

Theorem 3.2. *The functor* $\mathrm{lpc} : \mathbf{Top} \to \mathbf{LC}_0$ *is right adjoint to the inclusion functor* $\mathbf{LC}_0 \to \mathbf{Top}$.

Proof. We've already confirmed that we have all the right ingredients. Let's just put them together. First, we check that lpc is a functor. We have left to see what it does to morphisms. If $f : X \to Y$ is a continuous function of any spaces, then we may compose it with the continuous identity $\mathrm{lpc}(X) \to X$ to get a continuous function $f : \mathrm{lpc}(X) \to Y$. Since $\mathrm{lpc}(X)$ is locally path connected, Lemma 2.4 guarantees that $\mathrm{lpc}(f) : \mathrm{lpc}(X) \to \mathrm{lpc}(Y)$ is continuous (notice this is actually the same function, it's just the spaces have different topologies). Thus lpc is the identity on both underlying sets and functions. From here it is more or less obvious that lpc preserves identities and composition.

Now that we know lpc is a functor, Theorem 2.7 implies that lpc is, in fact, right adjoint to the inclusion $\mathbf{LC}_0 \to \mathbf{Top}$. □

The fact that lpc is a true coreflection functor is why we choose to refer to $\text{lpc}(X)$ as the *locally path-connected coreflection* of X. These categorical properties are used in an important way in [2].

3.2. **colimits in $\mathbf{LC}_0$.** In the next two sections, we'll see how colimits and limits in the category $\mathbf{LC}_0$ are related to lpc.

A topological sum (or disjoint union) of locally path-connected spaces is locally path connected. We have the following as an immediate consequence.

Proposition 3.3. *If $X = \coprod_{j\in J} X_j$ is a topological disjoint union of the spaces X_j, then $\text{lpc}(X) = \coprod_{j\in J} \text{lpc}(X_j)$.*

In **Top**, coequalizers are constructed using quotient spaces. We show that this agrees with what happens in $\mathbf{LC}_0$.

Lemma 3.4. *Every quotient space of a locally path connected space is locally path connected.*

Proof. Let $q : X \to Y$ be a quotient map where X is locally path-connected. Let U be an open set in Y and C be a non-empty path-component of U. It suffices to check that C is open in Y. Since q is quotient, we need to check that $q^{-1}(C)$ is open in X. If $x \in q^{-1}(C) \subseteq q^{-1}(U)$, then there is an open, path connected neighborhood V such that $x \in V \subseteq q^{-1}(U)$. We claim that $V \subseteq q^{-1}(C)$. Let $v \in V$ and $\alpha : [0,1] \to V$ be a path from x to v. Then $q \circ \alpha : [0,1] \to q(V) \subseteq U$ is a path from $q(x) \in C$ to $q(v)$. Since C is the path component of $q(x)$ in U, the path $q \circ \alpha$ must have image entirely in C. Thus α has image in $q^{-1}(C)$. In particular, $\alpha(1) = v \in q^{-1}(C)$. □

By the Colimit Existence Theorem [8, §V.4], the colimit of any diagram $F : J \to$ **Top** is the coequalizer of a pair of parallel maps $\coprod_{u:j\to k} F(j) \to \coprod_{i\in J} F(i)$. Since coequalizers are quotient maps, the fact that coproducts and quotients of locally path-connected spaces are locally path-connected spaces, implies that any colimt of locally path-connected spaces in **Top** is locally path-connected.

Theorem 3.5. *Colimits in $\mathbf{LC}_0$ agree with those in* **Top***.*

3.3. **limits in $\mathbf{LC}_0$.** Limits of locally path connected spaces behave much differently than colimits. The category $\mathbf{LC}_0$ is not even closed under products.

Example 3.6. The two point discrete space $X = \{0,1\}$ is locally path connected. However, $X^{\mathbb{N}} = \prod_{\mathbb{N}} X$ is homeomorphic to the Cantor set, which is not locally path connected.

The following is a convenient exception. Usually this proof can be found as an exercise in topology textbooks so we omit the proof.

Lemma 3.7. *A direct product of path-connected, locally path-connected spaces is locally path connected.*

Thus categorical products in $\mathbf{LC}_0$ agree with those in **Top**. Recall that the equalizer of parallel maps $f, g : X \to Y$ in **Top** the subspace $E_{f,g} = \{x \in X \mid f(x) = g(x)\}$ of X.

Example 3.8. Here's we'll see an example of an equalizer of maps on locally path connected spaces that is not locally path connected. Recall that $C_{1/n} \subseteq \mathbb{R}^2$ is the circle of radius $\frac{1}{n}$ centered at $(0, \frac{1}{n})$ so that $\mathbb{E}_1 = \bigcup_{n\in\mathbb{N}} C_{1/n}$ is the one-dimensional earring space (see Figure 3). Let $\ell_n : I \to \mathbb{E}_1$ be a loop based at $e_0 = (0, 0)$ that parameterizes the n-the circle $C_{1/n}$. Taking $K \subseteq I$ to be the ternary Cantor set, let $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ be an enumeration of the connected components of $I\backslash K$. Define $f : I \to \mathbb{E}_1$ map $f(K) = e_0$ and so that $f|_{[a_n,b_n]}$ is a reparameterization of ℓ_{2n-1}. Similarly, define $g : I \to \mathbb{E}_1$ so that $g(K) = e_0$ and $g|_{[a_n,b_n]}$ is a reparameterization of ℓ_{2n}. Thus f and g are loops in $\mathbb{E}_1$. They agree on the Cantor set but otherwise, f maps to the odd loops of $\mathbb{E}_1$ and g maps to the even loops of $\mathbb{E}_1$. We conclude that the equalizer of f and g in **Top** is $\{t \in I \mid f(t) = g(t)\} = K$ and certainly the middle-third Cantor set is not locally path connected.

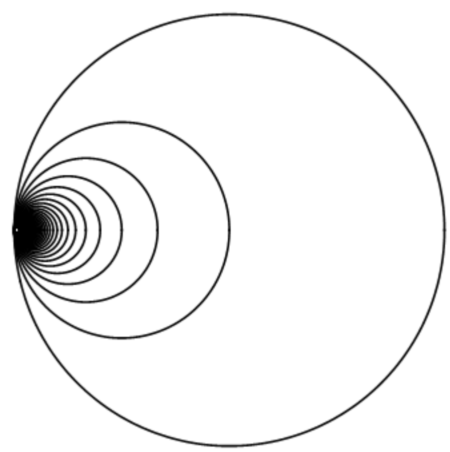

FIGURE 3. The one-dimensional earring space $\mathbb{E}_1$.

Proposition 3.9. *The equalizer of maps $f, g : X \to Y$ in $\mathbf{LC}_0$ is $\mathrm{lpc}(i) : \mathrm{lpc}(E_{f,g}) \to X$ where $E_{f,g} = \{x \in X \mid f(x) = g(x)\}$ and the inclusion $i : E_{f,g} \to X$ give the equalizer of f and g in* **Top**.

Proof. Suppose $f, g : X \to Y$ are parallel morphisms in $\mathbf{LC}_0$, i.e. of locally path connected spaces. Let $i : E_{f,g} \to X$ be the inclusion and $\mathrm{lpc}(i) : \mathrm{lpc}(E_{f,g}) \to \mathrm{lpc}(X) = X$ be the coreflection. Suppose $Z \in \mathbf{LC}_0$ and $k : Z \to X$ is a map such that $f \circ k = g \circ k$. Since $E_{f,g}$ is the equalizer in **Top**, there is a unique map $m : Z \to E_{f,g}$ such that $i \circ m = k$. Since Z is locally path connected, $\mathrm{lpc}(m) : Z \to \mathrm{lpc}(E_{f,g})$ is continuous by Lemma 2.4. Since $i = \mathrm{lpc}(i)$ as functions, we have $\mathrm{lpc}(i) \circ \mathrm{lpc}(m) = k$. The uniqueness of $\mathrm{lpc}(m)$ follows from that of m.

$$\begin{tikzcd} \mathrm{lpc}(E_{f,g}) \arrow[r, "id"] & E_{f,g} \arrow[r, "i"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ & Z \arrow[ul, dashed, "\mathrm{lpc}(m)"] \arrow[u, dashed, "m"] \arrow[ur, "k"'] & & \end{tikzcd}$$

□

The Limit Existence Theorem now implies the following. In short, it says that to take a limit in $\mathbf{LC}_0$, one must first take the limit in **Top** and then apply lpc in the case that this limit is not already locally path connected.

Theorem 3.10. *The limit of a diagram* $F : J \to \mathbf{LC}_0$ *is* $\mathrm{lpc}(\lim k \circ F)$ *where* $k : \mathbf{LC}_0 \to \mathbf{Top}$ *is the inclusion functors and* $\lim k \circ F$ *is the limit in* **Top**.

Example 3.11. If $f_{n+1,n} : X_{n+1} \to X_n$ is an inverse system in **Top** where each space X_n is locally path-connected, the inverse limit of this system in **Top** is the subspace

$$\varprojlim_n (X_n, f_{n+1,n}) = \{(x_n) \in \prod_{n\in\mathbb{N}} X_n \mid \forall n \in \mathbb{N},\, f_{n+1,n}(x_{n+1}) = x_n\},$$

of $\prod_{n\in\mathbb{N}} X_n$. Although $\prod_{n\in\mathbb{N}} X_n$ is locally path-connected, $\varprojlim_n (X_n, f_{n+1,n})$ often fails to be locally path connected. Therefore, $\mathrm{lpc}(\varprojlim_n (X_n, f_{n+1,n}))$ is the limit of the inverse system $(X_n, f_{n+1,n})$ in $\mathbf{LC}_0$.

It is worth noting that if each space X_n is also path connected, the limit $\varprojlim_n (X_n, f_{n+1,n})$ often fails to be path connected as well.

4. Algebraic Topology

In Example 2.11, we saw that lpc may change the homotopy type of a space. Despite this fact, we will see in this section that lpc does preserve homotopy equivalence between spaces and many other homotopy invariant properties (e.g. weak homotopy equivalence).

Theorem 4.1. *If* $f : X \to Y$ *and* $g : Y \to X$ *are (based) homotopy inverses, then* $\mathrm{lpc}(f) : \mathrm{lpc}(X) \to \mathrm{lpc}(Y)$ *and* $\mathrm{lpc}(g) : \mathrm{lpc}(Y) \to \mathrm{lpc}(X)$ *are also (based) homotopy inverses.*

Proof. Suppose $H : X \times I \to X$ is a homotopy from id_X to $g \circ f$ and $G : Y \times I \to Y$ is a homotopy from id_Y to $f \circ g$. Since lpc preserves products, the identity function $\mathrm{lpc}(X \times I) \to \mathrm{lpc}(X) \times \mathrm{lpc}(I) = \mathrm{lpc}(X) \times I$ is a homeomorphism. That is, $\mathrm{lpc}(X \times I) = \mathrm{lpc}(X) \times I$ as spaces. With this observation made and recalling that lpc is the identity on underlying functions, it is clear that $\mathrm{lpc}(H) : \mathrm{lpc}(X) \times I \to \mathrm{lpc}(X)$ is a homotopy from $id_{\mathrm{lpc}(X)}$ to $\mathrm{lpc}(g) \circ \mathrm{lpc}(f)$. Similarly, $\mathrm{lpc}(G) : \mathrm{lpc}(Y) \times I \to \mathrm{lpc}(Y)$ is a homotopy from $id_{\mathrm{lpc}(Y)}$ to $\mathrm{lpc}(f) \circ \mathrm{lpc}(g)$.

Moreover, note that if H and G are basepoint-preserving homotopies, then so are $\mathrm{lpc}(H)$ and $\mathrm{lpc}(G)$ (since they are equal to H and G as functions). □

Theorem 4.2. *If* $X \simeq Y$*, then* $\mathrm{lpc}(X) \simeq \mathrm{lpc}(Y)$.

For based spaces (X, x) and (Y, y), let $[(Y, y), (X, x)]$ denote the set of based homotopy classes of based maps $(Y, y) \to (X, x)$.

Theorem 4.3. *If Y is locally path-connected, the identity function $id : \mathrm{lpc}(X) \to X$ induces a bijection of homotopy classes $[(Y,y),(\mathrm{lpc}(X),x)] \to [(Y,y),(X,x)]$.*

Proof. Surjectivity follows directly from Lemma 2.4. Suppose $f,g : (Y,y) \to (\mathrm{lpc}(X),x)$ are maps such that $f,g : (Y,y) \to (X,x)$ are homotopic. Then $Y \times I$ is locally path-connected and the homotopy $H : Y \times [0,1] \to X$ is also continuous with respect to the topology of $\mathrm{lpc}(X)$. Thus we obtain a based homotopy $H : Y \times [0,1] \to \mathrm{lpc}(X)$ between $f,g : (Y,y) \to (\mathrm{lpc}(X),x)$. This shows the function on homotopy classes is injective. □

In the case that $Y = S^0$ is the two-point space, we see that $\mathrm{lpc}(X) \to X$ induces a bijection $\pi_0(\mathrm{lpc}(X)) \to \pi_0(X)$ of path components. When $Y = S^n$ is the n-sphere, we get the following corollary.

Corollary 4.4. *The identity function $id : \mathrm{lpc}(X) \to X$ induces an isomorphism $\pi_n(\mathrm{lpc}(X),x) \to \pi_n(X,x)$ of homotopy groups for all $n \geqslant 1$ and $x \in X$.*

Replacing maps on spheres with maps on the standard n-simplex Δ_n, we see there is a canonical bijection between singular n-chains in X and $\mathrm{lpc}(X)$. This means similar arguments give the same result for homology groups.

Corollary 4.5. *The identity function $id : \mathrm{lpc}(X) \to X$ induces isomorphisms $H_n(\mathrm{lpc}(X)) \to H_n(X)$ and $H^n(X) \to H^n(\mathrm{lpc}(X))$ of singular homology and cohomology groups for all $n \geqslant 0$.*

One of the limitations of standard methods in algebraic topology is that most techniques do not apply to non-locally path-connected spaces. For instance, covering spaces of locally path-connected spaces are uniquely determined (up to isomorphism) by the corresponding π_1 action on the fiber, but this convenience only translates to very special types of non-locally path-connected spaces. As long as the goal is to understand the homotopy and (co)homology groups of the space, and not to characterize the homotopy type, the lpc-coreflection allows one to assume the space in question is locally path-connected.

Definition 4.6. A space X is *semi-locally simply connected* if for every point $x \in X$, there is an open neighborhood U of x such that the inclusion $U \to X$ induces the trivial homomorphism $\pi_1(U,x) \to \pi_1(X,x)$ on fundamental groups.

Proposition 4.7. *A space X is semi-locally simply connected if and only if* $\mathrm{lpc}(X)$ *is semi-locally simply connected.*

Proof. First suppose X is semi-locally simply connected. Suppose $x \in X$ and U is an open neighborhood U of x such that the inclusion $U \to X$ induces the trivial homomorphism $\pi_1(U,x) \to \pi_1(X,x)$. Let C be the path component of x in U. Then C is an open neighborhood of x in $\mathrm{lpc}(X)$. The inclusion $f : C \to X$ induces a homomorphism $j_* : \pi_1(C,x) \to \pi_1(\mathrm{lpc}(X),x) \cong \pi_1(X,x)$ which factors as $\pi_1(C,x) \to \pi_1(U,x) \to \pi_1(X,x)$ where the later homomorphism is trivial. Thus j_* is trivial.

Conversely, suppose $\mathrm{lpc}(X)$ is semi-locally simply connected and $x \in X$. Find an open neighborhood C of x in $\mathrm{lpc}(X)$ such that $\pi_1(C,x) \to \pi_1(\mathrm{lpc}(X),x)$. We can assume C is a basic neighborhood, so that C is the path component of an open set U of X. If $\alpha : [0,1] \to U$ is a loop based x, then it must have image in C. Since α is null-homotopic in $\mathrm{lpc}(X)$, it must be null-homotopic when viewed as a loop in X. Thus $\pi_1(U,x) \to \pi_1(X,x)$ is trivial. □

It's an important fact from covering space theory that every path-connected, locally path-connected and semi-locally simply connected X admits a universal (simply connected) covering $p : \widetilde{X} \to X$.

Corollary 4.8. *If X is path connected and semilocally simply connected, then* $\mathrm{lpc}(X)$ *admits a universal space* $\widetilde{\mathrm{lpc}(X)}$ *and universal covering map* $p : \widetilde{\mathrm{lpc}(X)} \to \mathrm{lpc}(X)$.

The lpc-coreflection is perhaps even more useful when X does *not* admit a universal covering space.

Definition 4.9. A map $p : E \to X$ is a lpc-*lifting map* if

(1) E is non-empty, path connected, and locally path connected,
(2) for every based map $f : (Z, z) \to (X, x)$ from a path-connected, locally path-connected space Z and point $e \in p^{-1}(x)$ such that $f_{\#}(\pi_1(Z, z)) \subseteq p_{\#}(\pi_1(E, e))$, then there exists a unique map $\widetilde{f} : (Z, z) \to (E, e)$ such that $p \circ \widetilde{f} = f$.

We will call p a *generalized covering map* if E is also locally path connected. If E is locally path connected and simply connected, then p is a *generalized universal covering map.*

In [2], these kinds of generalizations of covering maps were defined as a specific case of a more general framework involving coreflective subcategories.

One problem with the notion of lpc-lifting maps is that they do not have a nice classification up to homeomorphism. Indeed, it's hard to check that for any path connected, non-locally path-connected space X, the identity function $id : \mathrm{lpc}(X) \to X$ is a unique-lifting map, which is not a homeomorphism.

However, generalized covering maps *are* classified up to homeomorphism in the same way covering maps are because the total space E is assumed to be in the category of spaces that p has unique lifting with respect to (namely path-connected, locally path-connected spaces). Specifically, given two generalized covering maps $p : (E, e) \to (X, x)$ and $p' : (E', e') \to (X, x)$, there is a unique homeomorphism $h : (E, e) \to (E', e')$ with $p' \circ h = p$ if and only if $p_{\#}(\pi_1(E, e)) = (p')_{\#}(\pi_1(E', e'))$ in $\pi_1(X, x)$.

Example 4.10. Verifying the following implications is a nice exercise: for a map $p : E \to X$ of path-connected spaces, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) where.

(1) p is a (Hurewicz) fibration with totally path-disconnected fibers,
(2) p is a lpc-lifting map,
(3) p is a Serre fibration with totally path-disconnected fibers.

Like with other kinds of fibrations, lpc-lifting maps are defined purely in terms of lifting properties, they form a category with many nice properties. The locally path-connected coreflection always allows one to promote an lpc-lifting map to a generalized covering map.

Theorem 4.11. *If $p : E \to X$ is an* lpc-*lifting map, then $p : \mathrm{lpc}(E) \to X$ is a generalized covering map.*

Proof. By assumption, E is path connected and thus $\mathrm{lpc}(E)$ is path connected and locally path-connected. Suppose $f : (Z, z) \to (X, x)$ from a path-connected, locally path-connected space Z and point $e \in p^{-1}(x)$ such that $f_{\#}(\pi_1(Z, z)) \subseteq$

$p_{\#}(\pi_1(\mathrm{lpc}(E), e))$. Since the identity $id : \mathrm{lpc}(E) \to E$ induces an isomorphism on fundamental groups, we have $p_{\#}(\pi_1(E, e)) = p_{\#}(\pi_1(\mathrm{lpc}(E), e))$. Since $f_{\#}(\pi_1(Z, z)) \subseteq p_{\#}(\pi_1(E, e))$ and $p : E \to X$ is an lpc-lifting map, there exists a unique map $\widetilde{f} : (Z, z) \to (E, e)$ such that $p \circ \widetilde{f} = f$. Since Z is locally path-connected, Lemma 2.4 gives that $\widetilde{f} : Z \to \mathrm{lpc}(E)$ is continuous. Since this is the same underlying function, uniqueness is clear and $p \circ \widetilde{f} = f$ still holds for $\widetilde{f} : Z \to \mathrm{lpc}(E)$. □

An inverse limit of path-connected spaces need not be path-connected. Hence, if we have a diagram of maps $f_j : E_j \to X_j$, $j \in J$ of path-connected spaces then the limit map $f = \lim_j f_j : \lim_j E_j \to \lim_j X_j$ (formally this is a limit taken in the arrow category of **Top**), then neither $E = \lim_j E_j$ nor $\lim_j X_j$ need to be path-connected when they are non-empty. Since we wish for our spaces to be path-connected (as we are looking at applications to algebraic topology), the standard approach is to instead, used based maps. Pick $e_0 \in \lim_j E_j$ and let E_0 be the path component of e_0 in E. Let $X_0 = p(E_0) \subseteq \lim_j X_j$ and $x_0 = p(e_0)$. Let $f_0 : E_0 \to X_0$ be the restriction of f to E_0. Now, if e_j is the projection of e_0 in E_j and $x_j = f_j(e_j)$, the map $f_0 : (E_0, e_0) \to (X, x_0)$ is the inverse limit of the diagram of based maps $f_j : (E_j, e_j) \to (X_j, x_j)$ in the arrow category of consisting of based maps of path-connected spaces. This reasoning implies the following.

Theorem 4.12. *Based* lpc*-covering maps are closed under forming arbitrary limits.*

Since we can apply the lpc-coreflection, this implies generalized covering maps are also closed under forming limits in their arrow category. However, unless the limit is a direct product, one must take the ordinary limit first and then apply lpc.

Theorem 4.13. *Based generalized covering maps are closed under forming arbitrary limits.*

5. Metrizability and Other Topological properties

5.1. lpc **preserves metrizability.** If X is metrizable it is not entirely obvious that $\mathrm{lpc}(X)$ must also be metrizable. It turns out the answer is yes but that we might lose separability along the way. In this post, we'll walk through the details, which I learned from some unpublished notes of Greg Conner and David Fearnley [5].

Theorem 5.1. *If X is path connected and metrizable, then there is a metric inducing the topology of* $\mathrm{lpc}(X)$ *such that the identity function* $id : \mathrm{lpc}(X) \to X$ *is distance non-increasing.*

Proof. Suppose X is a space whose topology is induced by a metric d. Define a distance function ρ on $\mathrm{lpc}(X)$ as follows: For any path $\alpha : [0, 1] \to X$ and $t \in [0, 1]$, let

$$\ell_t(\alpha) = d(\alpha(0), \alpha(t)) + d(\alpha(t), \alpha(1)).$$

Observe that $d(\alpha(0), \alpha(1)) \leqslant \ell_t(\alpha)$ for any $t \in I$ by the triangle inequality. Now let

$$\ell(\alpha) = \sup\{\ell_t(\alpha) \mid t \in [0, 1]\}.$$

For points $a, b \in X$, we define our distance function as

$$\rho(a, b) = \inf\{\ell(\alpha) \mid \alpha \text{ is a path from } a \text{ to } b\}$$

Since $d(a, b) \leqslant \ell(\alpha)$ for any path α from a to b, we get that $d(a, b) \leqslant \rho(a, b)$ showing that the identity $\mathrm{lpc}(X) \to X$ is non-increasing.

It remains to check that ρ is a metric which induces the topology of $\mathrm{lpc}(X)$. Some notation first: If α, β are paths in X such that $\alpha(1) = \beta(0)$, then $\alpha^-(t) = \alpha(1-t)$ denotes the reverse of α and $\alpha \cdot \beta$ denotes the usual concatenation of paths:

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & 0 \leqslant t \leqslant 1/2 \\ \beta(2t-1) & 1/2 \leqslant t \leqslant 1 \end{cases}.$$

Notice that $\ell(\alpha) = \ell(\alpha^-)$ and given $t \in [0, 1/2]$, we have

$$\begin{aligned} \ell_t(\alpha \cdot \beta) &= d(\alpha(0), \alpha(2t)) + d(\alpha(2t), \beta(1)) \\ &\leqslant d(\alpha(0), \alpha(2t)) + d(\alpha(2t), \alpha(1)) + d(\beta(0), \beta(1)) \\ &= \ell_{2t}(\alpha) + \ell_1(\beta) \\ &\leqslant \ell(\alpha) + \ell(\beta). \end{aligned}$$

Given $t \in [1/2, 1]$, we have

$$\begin{aligned} \ell_t(\alpha \cdot \beta) &= d(\alpha(0), \beta(2t-1)) + d(\beta(2t-1), \beta(1)) \\ &\leqslant d(\alpha(0), \alpha(1)) + d(\beta(0), \beta(2t-1)) + d(\beta(2t-1), \beta(1)) \\ &= \ell_0(\alpha) + \ell_{2t-1}(\beta) \\ &\leqslant \ell(\alpha) + \ell(\beta). \end{aligned}$$

Thus, $\ell(\alpha \cdot \beta) \leqslant \ell(\alpha) + \ell(\beta)$. Now we can check that ρ is a metric.

If $a = b$, then we may take α to be the constant path at this point. Then $\ell(\alpha) = 0$ showing $\rho(a, b) = 0$. Conversely, if $a \neq b$, consider any path $\alpha : [0, 1] \to X$ from a to b. Find $0 < t < 1$ such that $\alpha(t) \notin \{a, b\}$. Then $0 < d(a, b) \leqslant \ell_t(\alpha) \leqslant \ell(\alpha)$. Since α was arbitrary, we have $\rho(a, b) > 0$. Symmetry $\rho(a, b) = \rho(b, a)$ is clear since for every path α from a to b, there is a unique reverse path α^- from b to a with $\ell(\alpha) = \ell(\alpha^-)$. Suppose $a, b, c \in X$. Let α be any path from a to b and β be any path from b to c. Then there is a path $\alpha \cdot \beta$ is a path from a to c such that $\ell(\alpha \cdot \beta) \leqslant \ell(\alpha) + \ell(\beta)$. Therefore, $\rho(a, c) \leqslant \rho(a, b) + \rho(b, c)$ finishing the proof that ρ is a metric.

First, we show the metric topology induced by ρ is finer than the topology of $\mathrm{lpc}(X)$. Suppose U is an open set in X (with the topology induced by d) and C is some path component of U. Let $x \in C$. Find an ϵ-ball such that $B_d(x, \epsilon) \subseteq U$. We claim that $B_\rho(x, \epsilon) \subseteq C$: if $y \in B_\rho(x, \epsilon)$, then $\rho(x, y) < \epsilon$ so there is a path $\alpha : [0, 1] \to X$ from x to y such that $\ell(\alpha) < \epsilon$. Since $d(x, \alpha(t)) \leqslant \ell_t(\alpha) \leqslant \ell(\alpha) < \epsilon$ for all $t \in [0, 1]$, we conclude that $\alpha(t) \in B_d(x, \epsilon) \subseteq U$ for all t. Since α has image in U, we must have $\alpha(1) = y \in C$, proving the claim.

For the other direction, suppose $B_\rho(x, \epsilon)$ is an ϵ-ball with respect to ρ. Pick a point $y \in B_\rho(x, \epsilon)$ and let $\delta = \epsilon - \rho(x, y)$. We claim that the path component of y in $B_d(y, \delta/4)$ is contained in $B_\rho(x, \epsilon)$. Let α be a path in $B_d(y, \delta/4)$ such that $\alpha(0) = y$. It suffices to check that $z = \alpha(1) \in B_\rho(x, \epsilon)$. Notice that $\ell_t(\alpha) = d(y, \alpha(t)) + d(\alpha(t), z) < \frac{\delta}{4} + \frac{\delta}{2} = \frac{3\delta}{4}$ for all $t \in [0, 1]$. Thus $\ell(\alpha) \leqslant \frac{3\delta}{4}$ showing that $\rho(y, z) < \delta$. We now have

$$\rho(x, z) \leqslant \rho(x, y) + \rho(y, z) < (\epsilon - \delta) + \delta = \epsilon$$

proving the claim. $\square$

5.2. **Separability.**

Example 5.2. One thing to be wary of is that $\mathrm{lpc}(X)$ can fail to be separable even if X is a compact metric space. For instance, let A be a Cantor set in $[1,2]$. Then we can use the construction of generalized wedges of circles in Example 2.11 to construct the planar set $X = C_A$ which is a compact metric space (and certainly separable). This is a wedge of circles where the circles are parameterized by a Cantor set. But $\mathrm{lpc}(X)$ is an uncountable wedge of circles (with a metric topology - not the CW topology - at the joining point) and this is not separable. The general problem here is that there might be open sets of X which have uncountably many path components.

For any given space Y, $\pi_0(Y)$ will denote the set of path components of Y.

Theorem 5.3. *Let X be a metric space. Then $\mathrm{lpc}(X)$ is separable if and only if X is separable and $\pi_0(U)$ is countable for every open set $U \subseteq X$.*

Proof. If $\mathrm{lpc}(X)$ is separable, then since the identity function $\mathrm{lpc}(X) \to X$ is continuous and surjective, X is separable as the continuous image of a separable space. Now pick a countable dense set $A \subset \mathrm{lpc}(X)$ and let U be a non-empty open set in X. Now $\pi_0(U)$ is the set of path components of U. If $C \in \pi_0(U)$, then C is open in $\mathrm{lpc}(X)$ and thus there is a point $a \in A \cap C$. This gives a surjection from a subset of A onto $\pi_0(U)$ showing that $\pi_0(U)$ is countable.

For the converse, if X is a separable metric space then it has a countable basis $\mathscr{B}$. Furthermore, we assume $\pi_0(B)$ is countable for every set $B \in \mathscr{B}$. Let $\mathscr{C} = \bigcup_{B\in\mathscr{B}} \pi_0(B)$ be the collection of all path components of the basic open sets. Then $\mathscr{C}$ is countable. If C is the path component of x in an open set U of X, then there is a $B \in \mathscr{B}$ such that $x \in B \subseteq U$. Now if D is the path component of x in B, then $x \in D \subseteq C$ where $D \in \mathscr{C}$. This shows $\mathscr{C}$ forms a countable basis for the topology of $\mathrm{lpc}(X)$. Since $\mathrm{lpc}(X)$ is metrizable (by Theorem 5.1), it is also separable. □

5.3. **Some other topological properties.**

Proposition 5.4. *Suppose X is Hausdorff but not locally path connected. Then $\mathrm{lpc}(X)$ is not compact.*

Proof. If $\mathrm{lpc}(X)$ is compact, then $id : \mathrm{lpc}(X) \to X$ is a homeomorphism by the Closed Mapping Theorem. □

Proposition 5.5. *A space X is first countable at $x \in X$ if and only if $\mathrm{lpc}(X)$ is first countable at x.*

Proof. One direction is clear since the topology of $\mathrm{lpc}(X)$ is finer than that of X. Suppose X is first countable at x. Let $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ be a neighborhood base at x in X. Let C_n be the path component of x in U_n. We check that $\{C_n\}_{n\in\mathbb{N}}$ is a neighborhood base at x in $\mathrm{lpc}(X)$. Let V be an open neighborhood of x in X. Let D be the path component of x in V so that D is a basic open neighborhood of x in $\mathrm{lpc}(X)$. Find n such that $U_n \subseteq V$. Then $C_n \subseteq D$. This proves $\{C_n\}_{n\in\mathbb{N}}$ is a neighborhood base as desired. □

5.4. **Topological groups.** Since lpc preserves products in the sense that $\mathrm{lpc}(X \times Y) = \mathrm{lpc}(X) \times \mathrm{lpc}(Y)$, it is reasonable to suspect that lpc will preserve algebraic operations on spaces, e.g. those of topological monoids, groups, rings, etc. We'll focus on the group case here but other situations follow with similar arguments.

Theorem 5.6. *If G is a topological group, then* $\mathrm{lpc}(G)$ *is also a topological group under the same operation.*

Proof. Since $\mathrm{lpc}(G)$ and G have the same underlying sets, we impart $\mathrm{lpc}(G)$ with the operation of G so that $\mathrm{lpc}(G)$ becomes a group. If inversion $i : G \to G$, $i(g) = g^{-1}$ is continuous, then so is the inversion function $\mathrm{lpc}(i) : \mathrm{lpc}(G) \to \mathrm{lpc}(G)$. Similarly, if $\mu : G \times G \to G$ is continuous, then we have $\mathrm{lpc}(G \times G) = \mathrm{lpc}(G) \times \mathrm{lpc}(G)$ and the coreflection of μ is the continuous operation $\mathrm{lpc}(\mu) : \mathrm{lpc}(G) \times \mathrm{lpc}(G) \to \mathrm{lpc}(G)$ for $\mathrm{lpc}(G)$. $\square$

Example 5.7. If $G = \varprojlim_n G_n$ is an inverse limit of discrete (or totally path-disconnected) groups, then G will be totally path-disconnected and thus $\mathrm{lpc}(G)$ is G with the discrete topology. If G is a Lie group, then $\mathrm{lpc}(G) = G$ since G is already locally path connected.

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