# Ramanujan-Slater Type Identities Related to the Moduli 18 and 24


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**Abstract**

We present several new families of Rogers-Ramanujan type identities related to the moduli 18 and 24. A few of the identities were found by either Ramanujan, Slater, or Dyson, but most are believed to be new. For one of these families, we discuss possible connections with Lie algebras. We also present two families of related false theta function identities.



*Key words:* Rogers-Ramanujan identities, Bailey pairs, $q$-series identities, basic hypergeometric series, false theta functions, affine Lie algebras, principal character
*1991 MSC:* 11B65, 33D15, 05A10, 17B57, 17B10


## 1 Introduction

The Rogers-Ramanujan identities are

**Theorem 1.1 (The Rogers-Ramanujan Identities)**

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty}, \tag{1.1}$$


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*and*

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}, \tag{1.2}$$

*where*

$$(a;q)_m = \prod_{j=0}^{m-1}(1-aq^j),$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j),$$

*and*

$$(a_1,a_2,\dots,a_r;q)_s = (a_1;q)_s(a_2;q)_s\dots(a_r;q)_s.$$

(Although the results in this paper may be considered purely from the point of view of formal power series, they also yield identities of analytic functions provided $|q|<1$.)

The Rogers-Ramanujan identities are due to L. J. Rogers [29], and were rediscovered independently by S. Ramanujan [28] and I. Schur [31]. Rogers and others discovered many series–product identities similar in form to the Rogers-Ramanujan identities, and such identities are called "identities of the Rogers-Ramanujan type." Two of the largest collections of Rogers-Ramanujan type identities are contained in Slater's paper [34] and Ramanujan's Lost Notebook [8, Chapters 10–11], [9, Chapters 1–5].

Rogers-Ramanujan type identities occur in closely related "families." Just as there are two Rogers-Ramanujan identities related to the modulus 5, there are a family of three Rogers-Selberg identities related to the modulus 7 [30, p. 331, (6)], a family of three identities related to the modulus 9 found by Bailey [11, p. 422, Eqs. (1.6)–(1.8)], a family of four identities related to the modulus 27 found by Dyson [11, p. 433, Eqs. (B1)–(B4)], etc.

While both Ramanujan and Slater usually managed to find all members of a given family, this was not always the case. In this paper, we present several complete families of identities for which Ramanujan or Slater found only one member, as well as two complete new families.

The following family of four identities related to the modulus 18 is believed to be new:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}} = \frac{(q,q^8,q^9;q^9)_\infty(q^7,q^{11};q^{18})_\infty}{(q;q)_\infty} \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}} = \frac{(q^2,q^7,q^9;q^9)_\infty(q^5,q^{13};q^{18})_\infty}{(q;q)_\infty} \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q;q)_n(q;q)_{2n+1}} = \frac{(q^3,q^6,q^9;q^9)_\infty(q^3,q^{15};q^{18})_\infty}{(q;q)_\infty} \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3;q^3)_n}{(q^2;q^2)_n(q^{n+2};q)_{n+1}} = \frac{(q^4,q^5,q^9;q^9)_\infty(q,q^{17};q^{18})_\infty}{(q;q)_\infty} \tag{1.6}$$

**Remark 1.2** *We included Identity* (1.5) *in our joint paper with D. Bowman [13, Eq. (6.30)], as it also occurs as part of a different family of four identities.*

A closely related family of mod 18 identities is as follows.

$$1+\sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}(2+q^n)}{(q;q)_{n-1}(q;q)_{2n}} = \frac{(-q,-q^8,q^9;q^9)_\infty(q^7,q^{11};q^{18})_\infty}{(q;q)_\infty} \tag{1.7}$$

$$1+\sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}(1+2q^n)}{(q;q)_{n-1}(q;q)_{2n}} = \frac{(-q^2,-q^7,q^9;q^9)_\infty(q^5,q^{13};q^{18})_\infty}{(q;q)_\infty} \tag{1.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} = \frac{(-q^3,-q^6,q^9;q^9)_\infty(q^3,q^{15};q^{18})_\infty}{(q;q)_\infty} \tag{1.9}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^3)_n}{(q;q)_n^2(q^{n+2};q)_{n+1}} = \frac{(-q^4,-q^5,q^9;q^9)_\infty(q,q^{17};q^{18})_\infty}{(q;q)_\infty} \tag{1.10}$$

Identity (1.9) is due to Dyson [11, p. 434, Eq. (B3)] and also appears in Slater [34, p. 161, Eq. (92)]. In both [11] and [34], the right hand side of (1.9) appears in a different form and thus is seen to be a member of a different family of four identities related to the modulus 27.

Following Ramanujan (cf. [8, p. 11, Eq (1.1.7)]), let us use the notation

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Ramanujan recorded the identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}} = \frac{(q^2,q^{10},q^{12};q^{12})_\infty(q^8,q^{16};q^{24})_\infty}{\psi(-q)} \tag{1.11}$$

in his lost notebook [9, Entry 5.3.8]. As we see below, it is actually only one of a family of five similar identities.

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n} = \frac{(q,q^{11},q^{12};q^{12})_\infty(q^{10},q^{14};q^{24})_\infty}{\psi(-q)} \tag{1.12}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n} = \frac{(q^3,q^9,q^{12};q^{12})_\infty(q^6,q^{18};q^{24})_\infty}{\psi(-q)} \tag{1.13}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3;q^6)_n}{(q;q)_{2n+1}(-q;q)_{2n}} = \frac{(q^4,q^8,q^{12};q^{12})_\infty(q^4,q^{20};q^{24})_\infty}{\psi(-q)} \tag{1.14}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_{n+1}(-q^6;q^6)_n}{(q^4;q^4)_n(q^{2n+4};q^2)_{n+1}} = \frac{(q^5,q^7,q^{12};q^{12})_\infty(q^2,q^{22};q^{24})_\infty}{\psi(-q)} \tag{1.15}$$

Ramanujan also recorded the identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(q^3;q^6)_n}{(q;q^2)_n^2(q^4;q^4)_n} = \frac{(-q^2,-q^{10},q^{12};q^{12})_\infty(q^8,q^{16};q^{24})_\infty}{\psi(-q)} \tag{1.16}$$

in the lost notebook [9, Entry 5.3.9].

Again, it is one of a family of five similar identities. This time, however, two of the remaining four identities were found by Slater. Identity (1.19) is a corrected presentation of [34, p. 164, Eq. (110)] and identity (1.20) is a corrected presentation of [34, p. 163, Eq. (108)].

$$1+\sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}(2+q^{2n})}{(q^2;q^2)_{2n}(q^2;q^2)_{n-1}} = \frac{(-q,-q^{11},q^{12};q^{12})_\infty(q^{10},q^{14};q^{24})_\infty}{\psi(-q)} \tag{1.17}$$

$$1+\sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}(1+2q^{2n})}{(q^2;q^2)_{2n}(q^2;q^2)_{n-1}} = \frac{(-q^3,-q^9,q^{12};q^{12})_\infty(q^6,q^{18};q^{24})_\infty}{\psi(-q)} \tag{1.18}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}(q;q^2)_n} = \frac{(-q^4,-q^8,q^{12};q^{12})_\infty(q^4,q^{20};q^{24})_\infty}{\psi(-q)} \tag{1.19}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_{n+1}(q^6;q^6)_n}{(q^{2n+4};q^2)_{n+1}(q^2;q^2)_n^2} = \frac{(-q^5,-q^7,q^{12};q^{12})_\infty(q^2,q^{22};q^{24})_\infty}{\psi(-q)} \tag{1.20}$$

We believe that the following family of five identities has not previously appeared in the literature:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q;q)_{2n}(-q;q)_{2n+1}(-q;q^2)_n} = \frac{(q,q^{11},q^{12};q^{12})_\infty(q^{10},q^{14};q^{24})_\infty}{\varphi(-q^2)} \tag{1.21}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^6)_n(-q^2;q^2)_n}{(q^2;q^2)_{2n}(-1;q^2)_n} = \frac{(q^2,q^{10},q^{12};q^{12})_\infty(q^8,q^{16};q^{24})_\infty}{\varphi(-q^2)} \tag{1.22}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n} = \frac{(q^3,q^9,q^{12};q^{12})_\infty(q^6,q^{18};q^{24})_\infty}{\varphi(-q^2)} \tag{1.23}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^6;q^6)_n}{(q^2;q^2)_{2n+1}} = \frac{(q^4,q^8,q^{12};q^{12})_\infty(q^4,q^{20};q^{24})_\infty}{\varphi(-q^2)} \tag{1.24}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n} = \frac{(q^5,q^7,q^{12};q^{12})_\infty(q^2,q^{22};q^{24})_\infty}{\varphi(-q^2)}, \tag{1.25}$$

where

$$\varphi(q) := \frac{(-q;-q)_\infty}{(q;-q)_\infty}$$

is another notation used by Ramanujan.

In the following counterpart to the preceding family, two of the five identities appear in Slater's list.

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(q^3;q^6)_n}{(q;q)_{2n+1}(-q;q)_{2n}(q;q^2)_n} = \frac{(-q,-q^{11},q^{12};q^{12})_\infty(q^{10},q^{14};q^{24})_\infty}{\varphi(-q^2)} \tag{1.26}$$

$$1+\sum_{n=1}^{\infty} \frac{q^{n(n+1)}(q^6;q^6)_{n-1}(-q^2;q^2)_n}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}} = \frac{(-q^2,-q^{10},q^{12};q^{12})_\infty(q^8,q^{16};q^{24})_\infty}{\varphi(-q^2)} \tag{1.27}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(q^3;q^6)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n} = \frac{(-q^3,-q^9,q^{12};q^{12})_\infty(q^6,q^{18};q^{24})_\infty}{\varphi(-q^2)} \tag{1.28}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^6;q^6)_n(-q^2;q^2)_n}{(q^2;q^2)_{2n+1}(q^2;q^2)_n} = \frac{(-q^4,-q^8,q^{12};q^{12})_\infty(q^4,q^{20};q^{24})_\infty}{\varphi(-q^2)} \tag{1.29}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)}(-q^2;q^2)_n(q^3;q^6)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n} = \frac{(-q^5,-q^7,q^{12};q^{12})_\infty(q^2,q^{22};q^{24})_\infty}{\varphi(-q^2)} \tag{1.30}$$

Identity (1.28) is due to Slater [34, p. 163, Eq. (107)]. Identity (1.29) is originally due to Dyson [11, p. 434, Eq. (D2)] and also appears in Slater [34, p. 160, Eq. (77)].

The following false theta series identities, which are closely related to identities (1.21)–(1.30), are believed to be new, except for (1.37) and (1.39). Identity (1.37) is due to Dyson [11, p. 434, Eq. (E1)], while Identity (1.39) appears in Ramanujan's lost notebook [9, Entry 5.4.2] and was independently rediscovered by Dyson [11, p. 434, Eq. (E2)].

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(-q^3;q^6)_n}{(q^{n+1};q)_n(-q^{n+1};q)_{n+1}(-q;q^2)_n}\\&\quad=\sum_{n=0}^{\infty}(-1)^n q^{18n^2+3n}(1+q^{30n+15}) - q\sum_{n=0}^{\infty}(-1)^n q^{18n^2+9n}(1+q^{18n+9})\end{aligned} \tag{1.31}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)}(-q^6;q^6)_n}{(q^{2n+4};q^2)_{n+1}(-q^2;q^2)_n} = \sum_{n=0}^{\infty}(-1)^n q^{18n^2+12n}(1+q^{12n+6}) \tag{1.32}$$

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(-q^3;q^6)_n}{(q^{2n+2};q^2)_{n+1}(-q;q^2)_n}\\&\quad=\sum_{n=0}^{\infty}(-1)^n q^{18n^2+3n}(1+q^{30n+15}) + q^3\sum_{n=0}^{\infty}(-1)^n q^{18n^2+15n}(1+q^{6n+3})\end{aligned} \tag{1.33}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(-q^6;q^6)_n}{(q^{2n+2};q^2)_{n+1}(-q^2;q^2)_n} = \sum_{n=0}^{\infty}(-1)^n q^{18n^2+6n}(1+q^{24n+12}) + 2q^4\sum_{n=0}^{\infty}(-1)^n q^{18n^2+18n} \quad (1.34)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)}(-q^3;q^6)_n}{(q^{2n+2};q^2)_{n+1}(-q;q^2)_n} = \sum_{n=0}^{\infty}(-1)^n q^{18n^2+9n}(1+q^{18n+9}) + q^2\sum_{n=0}^{\infty}(-1)^n q^{18n^2+15n}(1+q^{6n+3}) \quad (1.35)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(q^3;q^6)_n}{(q^{n+1};q)_{n+1}(-q^{n+1};q)_n(q;q^2)_n} = \sum_{n=0}^{\infty} q^{18n^2+3n}(1-q^{30n+15}) + q\sum_{n=0}^{\infty} q^{18n^2+9n}(1-q^{18n+9}) \quad (1.36)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)}(q^6;q^6)_n}{(q;q)_{2n+1}(-q;q)_{2n+2}} = \sum_{n=0}^{\infty} q^{18n^2+12n}(1-q^{12n+6}) \quad (1.37)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(q^3;q^6)_n}{(q^{2n+2};q^2)_{n+1}(q;q^2)_n} = \sum_{n=0}^{\infty} q^{18n^2+3n}(1-q^{30n+15}) - q^3\sum_{n=0}^{\infty} q^{18n^2+15n}(1-q^{6n+3}) \quad (1.38)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}(q^6;q^6)_n}{(q^2;q^2)_{2n+1}} = \sum_{n=0}^{\infty} q^{18n^2+6n}(1-q^{24n+12}) \quad (1.39)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)}(q^3;q^6)_n}{(q^{2n+2};q^2)_{n+1}(q;q^2)_n} = \sum_{n=0}^{\infty} q^{18n^2+9n}(1-q^{18n+9}) + q^2\sum_{n=0}^{\infty} q^{18n^2+15n}(1-q^{6n+3}) \quad (1.40)$$

In §2, we will review some standard definitions and results to be used in the sequel. In §3, we derive the Bailey pairs necessary to prove Identities (1.3)–(1.40). Proofs of Identities (1.3)–(1.30) will be provided in §4. In §5, we will discuss and prove the false theta series identities (1.31)–(1.40). Finally, in §6 we discuss possible connections between Identities (1.3)–(1.6) and the standard level 6 modules associated with the Lie algebra $A_2^{(2)}$.

## 2 Standard definitions and results

We will require a number of definitions and theorems from the literature. It will be convenient to adopt Ramanujan's notation for theta functions [8, p. 11, Eqs. (1.1.5)–(1.1.8)].

**Definition 2.1** *For $|ab| < 1$, let*

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{2.1}$$

$$\varphi(q) := f(q,q), \tag{2.2}$$

$$\psi(q) := f(q,q^3), \tag{2.3}$$

$$f(-q) := f(-q,-q^2). \tag{2.4}$$

Both the Jacobi triple product identity and the quintuple product identity were used extensively by Ramanujan (cf. [8], [9]) and Slater [34]. Rogers, on the other hand, appears to have been unaware of the quintuple product identity, since referred to [29, p. 333, Eq. (16)]

$$\frac{(q^2;q^2)_\infty}{(q^{30};q^{30})_\infty (q;q^5)_\infty (q^4;q^5)_\infty} = (q^{13};q^{30})_\infty (q^{17};q^{30})_\infty + q(q^7;q^{30})_\infty (q^{23};q^{30})_\infty, \tag{2.5}$$

which follows immediately from the quintuple product identity, as a "remarkable identity" after observing that both sides of (2.5) are equal to the same series. Accordingly, we have chosen the name "Ramanujan-Slater type identities" in our title for the identities in this paper rather than "Rogers-Ramanujan type identities."

Many proofs of the Jacobi triple product identity are known; see, e.g., [7, pp. 496–500] for two proofs. For a history and many proofs of the quintuple product identity, see S. Cooper's excellent survey article [19].

**Theorem 2.2 (Jacobi's triple product identity)** *For $|ab| < 1$,*

$$f(a,b) = (-a,-b,ab;ab)_\infty. \tag{2.6}$$

**Theorem 2.3 (Quintuple product identity)** *For $|w| < 1$ and $x \neq 0$,*

$$\begin{aligned} f(-wx^3, -w^2x^{-3}) + x f(-wx^{-3}, -w^2x^3) &= \frac{f(w/x, x) f(-w/x^2, -wx^2)}{f(-w^2)} \\ &= (-wx^{-1}, -x, w; w)_\infty (wx^{-2}, wx^2; w^2)_\infty. \end{aligned} \tag{2.7}$$

The following is a special case of Bailey's ${}_6\psi_6$ summation formula [10, Eq. (4.7)] which appears in Slater [33, p. 464, Eq. (3.1)].

**Theorem 2.4 (Bailey)**

$$\sum_{r=-\infty}^{\infty} \frac{(1-aq^{6r})(q^{-n};q)_{3r}(e;q^3)_r a^{2r} q^{3nr}}{(1-a)(aq^{n+1};q)_{3r}(aq^3/e;q^3)_r e^r}$$
$$= \frac{(a;q^3)_\infty (q^3/a;q^3)_\infty (aq^2/e;q^3)_\infty (aq/e;q^3)_\infty (q;q)_n (aq;q)_n (a^2/e;q^3)_n}{(q;q^3)_\infty (q^2;q^3)_\infty (q^3/e;q^3)_\infty (a^2/e;q^3)_\infty (a;q)_{2n} (aq/e;q)_n}, \quad (2.8)$$

*where $a$ must be a power of $q$ so that the series terminates below.*

The next two $q$-hypergeometric summation formulas are due to to Andrews [2, p. 526, Eqs. (1.8) and (1.9) respectively].

**Theorem 2.5 ($q$-analog of Gauss's ${}_2F_1(\frac{1}{2})$ sum)**

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(a;q^2)_n (b;q^2)_n}{(q^2;q^2)_n (abq^2;q^4)_n} = \frac{(aq^2;q^4)_\infty (bq^2;q^4)_\infty}{(q^2;q^4)_\infty (abq^2;q^4)_\infty}. \quad (2.9)$$

**Theorem 2.6 ($q$-analog of Bailey's ${}_2F_1(\frac{1}{2})$ sum)**

$$\sum_{n=0}^{\infty} \frac{(bq;q^2)_n (b^{-1}q;q^2)_n c^n q^{n^2}}{(cq;q^2)_n (q^4;q^4)_n} = \frac{(b^{-1}cq^2;q^4)_\infty (bcq^2;q^4)_\infty}{(cq;q^2)_\infty}. \quad (2.10)$$

**Definition 2.7** *A pair of sequences of rational functions*

$$(\{\alpha_n(a,q)\}_{n=0}^{\infty}, \{\beta_n(a,q)\}_{n=0}^{\infty})$$

*is called a* Bailey pair relative to $a$ *if*

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_r(a,q)}{(aq;q)_{n+r}(q;q)_{n-r}}. \quad (2.11)$$

Bailey [12, p. 3, Eq. (3.1)] proved a key result, now known as "Bailey's lemma," which led to the discovery of many Rogers-Ramanujan type identities.

We will require several special cases of Bailey's lemma.

**Theorem 2.8** *If $(\{\alpha_n(a,q)\}, \{\beta_n(a,q)\})$ form a Bailey pair, then*

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n(a,q) = \frac{1}{(aq;q)_\infty} \sum_{r=0}^{\infty} a^r q^{r^2} \alpha_r(a,q) \quad (2.12)$$

$$\sum_{n=0}^{\infty} a^n q^{n^2} (-q;q^2)_n \beta_n(a,q^2) = \frac{(-aq;q^2)_\infty}{(aq^2;q^2)_\infty} \sum_{r=0}^{\infty} a^r q^{r^2} \alpha_r(a,q^2) \quad (2.13)$$

$$\frac{1}{1-q^2} \sum_{n=0}^{\infty} q^{n(n+1)} (-q^2;q^2)_n \beta_n(q^2,q^2) = \frac{1}{\varphi(-q^2)} \sum_{r=0}^{\infty} q^{r(r+1)} \alpha_r(q^2,q^2). \quad (2.14)$$

$$\frac{1}{1-q^2}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}(q^2;q^2)_n\beta_n(q^2,q^2)=\sum_{r=0}^{\infty}(-1)^r q^{r(r+1)}\alpha_r(q^2,q^2). \quad (2.15)$$

Eq. (2.12) is [12, p. 3, Eq. (3.1) with $\rho_1, \rho_2 \to \infty$]. Eq. (2.13) is [12, p. 3, Eq. (3.1) with $\rho_1 = -\sqrt{q};\ \rho_2 \to \infty$]. Eq. (2.14) is [12, p. 3, Eq. (3.1) with $\rho_1 = -q;\ \rho_2 \to \infty$]. Eq. (2.15) is [12, p. 3, Eq. (3.1) with $\rho_1 = q;\ \rho_2 \to \infty$].

## 3 Specific Bailey pairs

**Lemma 3.1** *If*

$$\alpha_n(1,q)=\begin{cases} 1 & \text{if } n=0 \\ q^{\frac{9}{2}r^2-\frac{3}{2}r}(1+q^{3r}) & \text{if } n=3r>0 \\ -q^{\frac{9}{2}r^2-\frac{9}{2}r+1} & \text{if } n=3r-1 \\ -q^{\frac{9}{2}r^2+\frac{9}{2}r+1} & \text{if } n=3r+1 \end{cases}$$

*and*

$$\beta_n(1,q)=\frac{(-1;q^3)_n}{(q;q)_{2n}(-1;q)_n},$$

*then $(\alpha_n(1,q), \beta_n(1,q))$ form a Bailey pair relative to 1.*

**PROOF.** Set $a=q$ and $e=-q^2$ in (2.8) and simplify to obtain

$$\sum_{r\geqq 0}\frac{(1-q^{6r+1})q^{\frac{9}{2}r^2-\frac{3}{2}r}}{(q;q)_{n-3r}(q;q)_{n+3r+1}}=\frac{(-1;q^3)_n}{(q;q)_{2n}(-1;q)_n}. \quad (3.1)$$

$$\begin{aligned}
&\sum_{r=0}^{n}\frac{\alpha_r(1,q)}{(q;q)_{n-r}(q;q)_{n+r}} \\
&=\frac{1}{(q;q)_n^2}+\sum_{r\geqq 1}\frac{\alpha_{3r}(1,q)}{(q;q)_{n-3r}(q;q)_{n+3r}}+\sum_{r\geqq 1}\frac{\alpha_{3r-1}(1,q)}{(q;q)_{n-3r+1}(q;q)_{n+3r-1}} \\
&\quad+\sum_{r\geqq 0}\frac{\alpha_{3r+1}(1,q)}{(q;q)_{n-3r-1}(q;q)_{n-3r+1}} \\
&=\frac{1}{(q;q)_n^2}+\sum_{r\geqq 1}\frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}(1+q^{3r})}{(q;q)_{n-3r}(q;q)_{n+3r}}-\sum_{r\geqq 1}\frac{q^{\frac{9}{2}r^2-\frac{9}{2}r+1}}{(q;q)_{n-3r+1}(q;q)_{n+3r-1}} \\
&\quad-\sum_{r\geqq 0}\frac{q^{\frac{9}{2}r^2+\frac{9}{2}r+1}}{(q;q)_{n-3r-1}(q;q)_{n-3r+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r\in\mathbb{Z}} \frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}}{(q;q)_{n-3r}(q;q)_{n+3r}} - \sum_{r\in\mathbb{Z}} \frac{q^{\frac{9}{2}r^2+\frac{9}{2}r+1}}{(q;q)_{n+3r+1}(q;q)_{n-3r-1}} \\
&= \sum_{r\in\mathbb{Z}} \frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}}{(q;q)_{n-3r}(q;q)_{n+3r+1}} \Big((1-q^{n+3r+1}) - q^{6r+1}(1-q^{n-3r})\Big) \\
&= \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1})}{(q;q)_{n-3r}(q;q)_{n+3r+1}} \\
&= \frac{(-1;q^3)}{(q;q)_{2n}(-1;q)_n} \text{ (by (3.1))} . \qquad \square
\end{aligned}
$$

**Lemma 3.2** *If*

$$
\alpha_n(1,q) = \begin{cases} 1 & \text{if } n=0 \\ q^{\frac{9}{2}r^2-\frac{3}{2}r}(1+q^{3r}) & \text{if } n=3r>0 \\ -q^{\frac{9}{2}r^2-\frac{3}{2}r} & \text{if } n=3r-1 \\ -q^{\frac{9}{2}r^2+\frac{3}{2}r} & \text{if } n=3r+1 \end{cases}
$$

*and*

$$
\beta_n(1,q) = \frac{q^n(-1;q^3)_n}{(q;q)_{2n}(-1;q)_n},
$$

*then* $(\alpha_n(1,q), \beta_n(1,q))$ *form a Bailey pair relative to* 1.

**PROOF.**

$$
\begin{aligned}
&\sum_{r=0}^{n} \frac{\alpha_r(1,q)}{(q;q)_{n-r}(q;q)_{n+r}} \\
&= \frac{1}{(q;q)_n^2} + \sum_{r\geqq 1} \frac{\alpha_{3r}(1,q)}{(q;q)_{n-3r}(q;q)_{n+3r}} + \sum_{r\geqq 1} \frac{\alpha_{3r-1}(1,q)}{(q;q)_{n-3r+1}(q;q)_{n+3r-1}} \\
&\qquad + \sum_{r\geqq 0} \frac{\alpha_{3r+1}(1,q)}{(q;q)_{n-3r-1}(q;q)_{n-3r+1}} \\
&= \frac{1}{(q;q)_n^2} + \sum_{r\geqq 1} \frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}(1+q^{3r})}{(q;q)_{n-3r}(q;q)_{n+3r}} - \sum_{r\geqq 1} \frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}}{(q;q)_{n-3r+1}(q;q)_{n+3r-1}} \\
&\qquad - \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2+\frac{3}{2}r}}{(q;q)_{n-3r-1}(q;q)_{n-3r+1}} \\
&= \sum_{r\in\mathbb{Z}} \frac{q^{\frac{9}{2}r^2+\frac{3}{2}r}}{(q;q)_{n-3r}(q;q)_{n+3r}} - \sum_{r\in\mathbb{Z}} \frac{q^{\frac{9}{2}r^2+\frac{3}{2}r}}{(q;q)_{n-3r-1}(q;q)_{n+3r+1}} \\
&= \sum_{r\in\mathbb{Z}} \frac{q^{\frac{9}{2}r^2-\frac{3}{2}r+n}}{(q;q)_{n-3r}(q;q)_{n+3r+1}} q^{3r-n} \Big((1-q^{n+3r+1}) - (1-q^{n-3r})\Big)
\end{aligned}
$$

$$= \sum_{r \geqq 0} \frac{q^{\frac{9}{2}r^2 - \frac{3}{2}r + n}(1 - q^{6r+1})}{(q;q)_{n-3r}(q;q)_{n+3r+1}}$$

$$= \frac{q^n(-1;q^3)}{(q;q)_{2n}(-1;q)_n} \text{ (by (3.1))} . \qquad \Box$$

**Lemma 3.3** *If*

$$\alpha_n(q,q) = \begin{cases} q^{\frac{9}{2}r^2 + \frac{3}{2}r} & \text{if } n = 3r \\ q^{\frac{9}{2}r^2 - \frac{3}{2}r} & \text{if } n = 3r - 1 \\ -2q^{\frac{9}{2}r^2 + \frac{9}{2}r + 1} & \text{if } n = 3r + 1 \end{cases}$$

*and*

$$\beta_n(q,q) = \frac{(-q^3;q^3)_n}{(q^2;q)_{2n}(-q;q)_n},$$

*then* $(\alpha_n(q,q), \beta_n(q,q))$ *form a Bailey pair relative to* $q$.

**PROOF.** Set $a = q^2$ and $e = -q$ in (3.1) and simplify to obtain

$$\sum_{r \in \mathbb{Z}} \frac{(1 - q^{3r+1})(1+q)q^{\frac{9}{2}r^2 + \frac{3}{2}r}}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}} = \frac{(-q^3;q^3)_n}{(q^2;q)_{2n}(-q^2;q)_n}. \tag{3.2}$$

Thus,

$$\frac{(-q^3;q^3)_n}{(q^2;q)_{2n}(-q;q)_n}$$

$$= \sum_{r \in \mathbb{Z}} \frac{(1 - q^{3r+1})(1 + q^{n+1})q^{\frac{9}{2}r^2 + \frac{3}{2}r}}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}}$$

$$= \sum_{r \in \mathbb{Z}} \frac{q^{\frac{9}{2}r^2 + \frac{3}{2}r}}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}}(1 - q^{n+3r+2} - q^{3r+1} + q^{n+1})$$

$$= \sum_{r \in \mathbb{Z}} \frac{q^{\frac{9}{2}r^2 + \frac{3}{2}r}}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}} \left( (1 - q^{n+3r+2}) - q^{3r+1}(1 - q^{n-3r}) \right)$$

$$= \sum_{r \in \mathbb{Z}} \frac{q^{\frac{9}{2}r^2 + \frac{3}{2}r}(1 - q^{n+3r+2})}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}} - \sum_{r \in \mathbb{Z}} \frac{q^{\frac{9}{2}r^2 + \frac{9}{2}r + 1}(1 - q^{n-3r})}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}}$$

$$= \sum_{r \in \mathbb{Z}} \frac{q^{\frac{9}{2}r^2 + \frac{3}{2}r}(1 - q^{n+3r+2})}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}} - \sum_{r \geqq 0} \frac{q^{\frac{9}{2}r^2 + \frac{9}{2}r + 1}}{(q^2;q)_{n+3r+1}(q;q)_{n-3r-1}}$$

$$- \sum_{r \geqq 1} \frac{q^{\frac{9}{2}r^2 - \frac{9}{2}r + 1}}{(q^2;q)_{n+3r-2}(q;q)_{n-3r+2}}$$

$$= \sum_{r \in \mathbb{Z}} \frac{q^{\frac{9}{2}r^2 + \frac{3}{2}r}(1 - q^{n+3r+2})}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}} - \sum_{r \geqq 0} \frac{q^{\frac{9}{2}r^2 + \frac{9}{2}r + 1}}{(q^2;q)_{n+3r+1}(q;q)_{n-3r-1}}$$

$$
\begin{aligned}
&\quad - \sum_{r \geqq 1} \frac{q^{\frac{9}{2}r^2 - \frac{9}{2}r + 1}}{(q;q)_{n+3r-1}(q^2;q)_{n-3r+1}} \\
&= \frac{1}{(q;q)_n (q^2;q)_{n+1}} + \sum_{r \geqq 1} \frac{q^{\frac{9}{2}r^2 + \frac{3}{2}r}}{(q^2;q)_{n+3r+1}(q;q)_{n-3r}} \\
&\quad - 2 \sum_{r \geqq 0} \frac{q^{\frac{9}{2}r^2 + \frac{9}{2}r + 1}}{(q^2;q)_{n-3r-1}(q^2;q)_{n+3r+1}}. \qquad \square
\end{aligned}
$$

**Lemma 3.4** *If*

$$
\alpha_n(q,q) = \begin{cases} q^{\frac{9}{2}r^2} & \text{if } n = 3r \text{ or } n = 3r-1 \\ -q^{\frac{9}{2}r^2 + 3r + \frac{1}{2}}(1 + q^{3r + \frac{3}{2}}) & \text{if } n = 3r+1 \end{cases}
$$

*and*

$$
\beta_n(q,q) = \frac{(-q^{3/2};q^3)_n}{(q^2;q)_{2n}(-q^{1/2};q)_n},
$$

*then* $(\alpha_n(q,q), \beta_n(q,q))$ *form a Bailey pair relative to* $q$.

**PROOF.** The algebraic details are similar to those of the preceding lemmas, and therefore will be omitted. In this case, the key is to set $a = q^2$ and $e = -q^{5/2}$ in (3.1), and observe that $1 - q^{6r+2} = (1 - q^{n+3r+2}) - q^{6r+2}(1 - q^{n-3r})$. $\square$

**Lemma 3.5** *If*

$$
\alpha_n(q,q) = \begin{cases} q^{\frac{9}{2}r^2 + 3r} & \text{if } n = 3r \\ q^{\frac{9}{2}r^2 - 3r} & \text{if } n = 3r-1 \\ -q^{\frac{9}{2}r^2}(q^{6r+\frac{3}{2}} + q^{3r}) & \text{if } n = 3r+1 \end{cases}
$$

*and*

$$
\beta_n(q,q) = \frac{q^n(-q^{3/2};q^3)_n}{(q^2;q)_{2n}(-q^{1/2};q)_n},
$$

*then* $(\alpha_n(q,q), \beta_n(q,q))$ *form a Bailey pair relative to* $q$.

**PROOF.** We first recall, as was noted in the proof of Lemma 3.4, that setting $a = q^2$ and $e = -q^{5/2}$ in (3.1) gives that

$$
\sum_{r \in \mathbb{Z}} \frac{(1 - q^{6r+2})q^{\frac{9}{2}r^2}}{(q;q)_{n-3r}(q^2;q)_{n+3r+1}} = \frac{(-q^{3/2};q^3)_n}{(q^2;q)_{2n}(-q^{1/2};q)_n}. \tag{3.3}
$$

Next,

$$
\begin{aligned}
&\sum_{r=0}^{n} \frac{\alpha_r(q,q)}{(q;q)_{n-r}(q^2;q)_{n+r}} \\
&= \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n-3r}(q^2;q)_{n+3r}} + \sum_{r\geqq 1} \frac{q^{\frac{9}{2}r^2-3r}}{(q;q)_{n-3r+1}(q^2;q)_{n+3r-1}} \\
&\qquad\qquad - \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2}(q^{6r+\frac{3}{2}}+q^{3r})}{(q;q)_{n-3r-1}(q^2;q)_{n+3r+1}} \\
&= \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n-3r}(q^2;q)_{n+3r}} + \sum_{r\leqq -1} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n+3r+1}(q^2;q)_{n-3r-1}} \\
&\qquad - \sum_{r\leqq 0} \frac{q^{\frac{9}{2}r^2-6r+\frac{3}{2}}}{(q;q)_{n+3r-1}(q^2;q)_{n-3r+1}} + \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n-3r-1}(q^2;q)_{n+3r+1}} \\
&= \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n-3r}(q^2;q)_{n+3r}} + \sum_{r\leqq -1} \frac{q^{\frac{9}{2}r^2+3r}}{(q^2;q)_{n+3r}(q;q)_{n-3r}} \\
&\qquad - \sum_{r\leqq -1} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n+3r+2}(q^2;q)_{n-3r-2}} - \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n-3r-1}(q^2;q)_{n+3r+1}} \\
&= \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n-3r}(q^2;q)_{n+3r}} + \sum_{r\leqq -1} \frac{q^{\frac{9}{2}r^2+3r}}{(q^2;q)_{n+3r}(q;q)_{n-3r}} \\
&\qquad - \sum_{r\leqq -1} \frac{q^{\frac{9}{2}r^2+3r}}{(q^2;q)_{n+3r+1}(q;q)_{n-3r-1}} - \sum_{r\geqq 0} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n-3r-1}(q^2;q)_{n+3r+1}} \\
&= \sum_{r\in\mathbb{Z}} \frac{q^{\frac{9}{2}r^2+3r}}{(q;q)_{n-3r}(q^2;q)_{n+3r+1}} \Big( (1-q^{n+3r+2}) - (1-q^{n-3r}) \Big) \\
&= q^n \sum_{r\in\mathbb{Z}} \frac{q^{\frac{9}{2}r^2}(1-q^{6r+2})}{(q;q)_{n-3r}(q^2;q)_{n+3r+1}} \\
&= \frac{q^n(-q^{3/2};q^3)}{(q^2;q)_{2n}(-q^{1/2};q)_n} \ \text{(by (3.3))} . \qquad \square
\end{aligned}
$$

**Lemma 3.6** *If*

$$
\alpha_n(q,q) = \begin{cases} q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1}) & \text{if } n=3r \\ -q^{\frac{9}{2}r^2-\frac{9}{2}r+1}(1-q^{6r-1}) & \text{if } n=3r-1 \\ 0 & \text{if } n=3r+1 \end{cases}
$$

*and*

$$
\beta_n(q,q) = \frac{(1-q)(-1;q^3)_n}{(q;q)_{2n}(-1;q)_n},
$$

*then* $(\alpha_n(q,q),\beta_n(q,q))$ *form a Bailey pair relative to* $q$.

**PROOF.** From (3.1),

$$(1-q)\sum_{r\in\mathbb{Z}}\frac{(1-q^{6r+1})q^{\frac{9}{2}r^2-\frac{3}{2}r}}{(q;q)_{n-3r}(q;q)_{n+3r+1}}=\frac{(1-q)(-1;q^3)_n}{(q;q)_{2n}(-1;q)_n}. \tag{3.4}$$

Next,

$$\begin{aligned}
&\sum_{r=0}^{n}\frac{\alpha_r(q,q)}{(q;q)_{n-r}(q^2;q)_{n+r}}\\
&=\sum_{r\geqq 0}\frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1})}{(q;q)_{n-3r}(q^2;q)_{n+3r}}-\sum_{r\geqq 1}\frac{q^{\frac{9}{2}r^2-\frac{9}{2}r+1}(1-q^{6r-1})}{(q;q)_{n-3r+1}(q^2;q)_{n+3r-1}}\\
&=(1-q)\sum_{r\geqq 0}\frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1})}{(q;q)_{n-3r}(q;q)_{n+3r+1}}-\sum_{r\leq -1}\frac{q^{\frac{9}{2}r^2+\frac{9}{2}r+1}(1-q^{-6r-1})}{(q;q)_{n+3r+1}(q^2;q)_{n-3r-1}}\\
&=(1-q)\sum_{r\geqq 0}\frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1})}{(q;q)_{n-3r}(q;q)_{n+3r+1}}-\sum_{r\leq -1}\frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1})}{(q;q)_{n+3r+1}(q^2;q)_{n-3r-1}}\\
&=(1-q)\sum_{r\in\mathbb{Z}}\frac{q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1})}{(q;q)_{n-3r}(q;q)_{n+3r+1}}\\
&=\frac{(1-q)(-1;q^3)_n}{(q;q)_{2n}(-1;q)_n}\ (\text{by (3.4)})\ . \qquad \square
\end{aligned}$$

**Lemma 3.7** *If*

$$\alpha_n(q,q)=\begin{cases}(-1)^r q^{\frac{9}{2}r^2-\frac{3}{2}r}\dfrac{1-q^{6r+1}}{1-q} & \text{if } n=3r\\ (-1)^{r+1}q^{\frac{9}{2}r^2-\frac{9}{2}r+1}\dfrac{1-q^{6r-1}}{1-q} & \text{if } n=3r-1\\ (-1)^{r+1}q^{\frac{9}{2}r^2+\frac{3}{2}r+1}\dfrac{1-q^{6r+3}}{1-q} & \text{if } n=3r+1\end{cases}$$

*and*

$$\beta_n(q,q)=\begin{cases}1, & n=0,\\ \dfrac{(q^3;q^3)_{n-1}}{(q^2;q)_{2n-1}(q;q)_{n-1}}, & n\geqq 1,\end{cases}$$

*then* $(\alpha_n(q,q),\beta_n(q,q))$ *form a Bailey pair relative to* $q$.

**PROOF.** Let $a=q$ and $e=q^2$ in (2.8) to get that

$$\sum_{r\in\mathbb{Z}}\frac{(1-q^{6r+1})(-1)^r q^{\frac{9}{2}r^2-\frac{3}{2}r}}{(q;q)_{n-3r}(q^2;q)_{n+3r}}=\frac{(q^3;q^3)_{n-1}}{(q^2;q)_{2n-1}(q;q)_{n-1}}. \tag{3.5}$$

Also recall the Bailey pair J6 [34, p. 149]:

$$\sum_{r\geqq 0}\frac{(-1)^r q^{\frac{9}{2}r^2+\frac{3}{2}r}}{(q;q)_{n-3r}(q^4;q)_{n+3r}}\frac{1-q^{6r+3}}{1-q^3}=\frac{(q^3;q^3)_n}{(q^3;q)_{2n}(q;q)_n},$$

so that multiplying both sides by $1/(1-q^2)$ and replacing $n$ by $n-1$ gives

$$\sum_{r\geqq 0}\frac{(-1)^r q^{\frac{9}{2}r^2+\frac{3}{2}r}(1-q^{6r+3})}{(q;q)_{n-3r-1}(q^2;q)_{n+3r+1}}=\frac{(q^3;q^3)_{n-1}}{(q^2;q)_{2n-1}(q;q)_{n-1}}. \tag{3.6}$$

Next,

$$\begin{aligned}
&\sum_{r=0}^{n}\frac{\alpha_r(q,q)}{(q;q)_{n-r}(q^2;q)_{n+r}}\\
&=\sum_{r\geqq 0}\frac{(-1)^r q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1})}{(q;q)_{n-3r}(q^2;q)_{n+3r}(1-q)}+\sum_{r\geqq 1}\frac{(-1)^{r+1} q^{\frac{9}{2}r^2-\frac{9}{2}r+1}(1-q^{6r-1})}{(q;q)_{n-3r+1}(q^2;q)_{n+3r-1}(1-q)}\\
&\qquad+\sum_{r\geqq 0}\frac{(-1)^{r+1} q^{\frac{9}{2}r^2+\frac{3}{2}r+1}(1-q^{6r+3})}{(q;q)_{n-3r-1}(q^2;q)_{n+3r+1}(1-q)}\\
&=\frac{1}{1-q}\sum_{r\in\mathbb{Z}}\frac{(-1)^r q^{\frac{9}{2}r^2-\frac{3}{2}r}(1-q^{6r+1})}{(q;q)_{n-3r}(q^2;q)_{n+3r}}-\frac{q}{1-q}\sum_{r\geqq 0}\frac{(-1)^r q^{\frac{9}{2}r^2+\frac{3}{2}r}(1-q^{6r+3})}{(q;q)_{n-3r-1}(q^2;q)_{n+3r+1}}\\
&=\frac{(q^3;q^3)_{n-1}}{(q^2;q)_{2n-1}(q;q)_{n-1}}\ \text{(by (3.5) and (3.6))}\ . \qquad \square
\end{aligned}$$

## 4 Proofs of Identities (1.3)–(1.20)

**Theorem 4.1** *Identity* (1.3) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.2 into Eq. (2.12) with $a=1$ to obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-1;q^3)_n}{(q;q)_{2n}(-1;q)_n}\\
&=\frac{1}{(q;q)_\infty}\left(1+\sum_{r=1}^{\infty}q^{\frac{27}{2}r^2-\frac{3}{2}r}(1+q^{3r})-\sum_{r=1}^{\infty}q^{\frac{27}{2}r^2-\frac{15}{2}r+1}-\sum_{r=0}^{\infty}q^{\frac{27}{2}r^2+\frac{15}{2}r+1}\right)\\
&=\frac{1}{(q;q)_\infty}\left(\sum_{r=-\infty}^{\infty}q^{\frac{27}{2}r^2-\frac{3}{2}r}-q\sum_{r=-\infty}^{\infty}q^{\frac{27}{2}r^2-\frac{15}{2}r}\right)
\end{aligned}$$

$$= \frac{f(q^{12},q^{15}) - qf(q^6,q^{21})}{f(-q)}$$
$$= \frac{(q,q^8,q^9;q^9)_\infty (q^7,q^{11};q^{18})_\infty}{(q;q)_\infty} \qquad \text{(by (2.7))}\ . \qquad \square$$

**Theorem 4.2** *Identity* (1.4) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.1 into Eq. (2.12) with $a = 1$ to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^3)_n}{(q;q)_{2n}(-1;q)_n}$$
$$= \frac{1}{(q;q)_\infty}\left(1 + \sum_{r=1}^{\infty} q^{\frac{27}{2}r^2-\frac{3}{2}r}(1+q^{3r}) - \sum_{r=1}^{\infty} q^{\frac{27}{2}r^2-\frac{21}{2}r+2} - \sum_{r=0}^{\infty} q^{\frac{27}{2}r^2+\frac{21}{2}r+2}\right)$$
$$= \frac{1}{(q;q)_\infty}\left(\sum_{r=-\infty}^{\infty} q^{\frac{27}{2}r^2-\frac{3}{2}r} - q^2 \sum_{r=-\infty}^{\infty} q^{\frac{27}{2}r^2-\frac{21}{2}r}\right)$$
$$= \frac{f(q^{12},q^{15}) - q^2 f(q^3,q^{24})}{f(-q)}$$
$$= \frac{(q^2,q^7,q^9;q^9)_\infty (q^5,q^{13};q^{18})_\infty}{(q;q)_\infty} \qquad \text{(by (2.7))}\ . \qquad \square$$

**Theorem 4.3** *Identity* (1.5) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.3 into Eq. (2.12) with $a = q$ to obtain

$$\frac{1}{1-q}\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^3;q^3)_n}{(q^2;q)_{2n}(-q;q)_n}$$
$$= \frac{1}{(q;q)_\infty}\left(\sum_{r=-\infty}^{\infty} q^{\frac{27}{2}r^2-\frac{9}{2}r} - q^3 \sum_{r=-\infty}^{\infty} q^{\frac{27}{2}r^2-\frac{27}{2}r}\right)$$
$$= \frac{f(q^9,q^{18}) - q^3 f(1,q^{27})}{f(-q)}$$
$$= \frac{(q^3,q^6,q^9;q^9)_\infty (q^3,q^{15};q^{18})_\infty}{(q;q)_\infty} \qquad \text{(by (2.7))}\ . \qquad \square$$

**Theorem 4.4** *Identity* (1.6) *is valid.*

**PROOF.**

$$\frac{(q^4,q^5,q^9;q^9)_\infty (q,q^{17};q^{18})_\infty}{(q;q)_\infty}$$

$$
\begin{aligned}
&= \frac{f(q^6,q^{21}) - qf(q^3,q^{24})}{f(-q)} \\
&= \frac{f(q^{12},q^{15}) - q^2f(q^3,q^{24})}{qf(-q)} - \frac{f(q^{12},q^{15}) - qf(q^6,q^{21})}{qf(-q)} \\
&= q^{-1}\left(\sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}} - \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}}\right) \text{ (by (1.4) and (1.3))} \\
&= q^{-1}\left(\sum_{n=-1}^{\infty} \frac{q^{n^2+2n+1}(-1;q^3)_{n+1}}{(-1;q)_{n+1}(q;q)_{2n+2}} - \sum_{n=-1}^{\infty} \frac{q^{n^2+3n+2}(-1;q^3)_{n+1}}{(-1;q)_{n+1}(q;q)_{2n+2}}\right) \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3;q^3)_n(1-q^{n+1})}{(-q;q)_n(q;q)_{2n+2}} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3;q^3)_n}{(q^2;q^2)_n(q^{n+2};q)_{n+1}}. \qquad \square
\end{aligned}
$$

**Theorem 4.5** *Identity* (1.7) *is valid.*

**PROOF.**

$$
\begin{aligned}
&\frac{(-q,-q^8,q^9;q^9)_\infty(q^7,q^{11};q^{18})_\infty}{(q;q)_\infty} \\
&= \frac{f(-q^{12},-q^{15}) + qf(-q^6,-q^{21})}{f(-q)} \text{ (by (2.7))} \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}} + q\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}} \\
&\qquad \text{(by [11, p. 433, Eqs. (B4) and (B2)])} \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}} + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n}} \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n}} \Big((1-q^{2n}) + (1-q^n)\Big) \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n}} \left((1-q^n)\left(1+q^n+1\right)\right) \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}(2+q^n)}{(q;q)_{n-1}(q;q)_{2n}}. \qquad \square
\end{aligned}
$$

**Theorem 4.6** *Identity* (1.8) *is valid.*

**PROOF.**

$$
\frac{(-q^2,-q^7,q^9;q^9)_\infty(q^5,q^{13};q^{18})_\infty}{(q;q)_\infty}
$$

$$
\begin{aligned}
&= \frac{f(-q^{12},-q^{15})+q^2f(-q^3,-q^{24})}{f(-q)} \text{ (by (2.7))}\\
&= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}}+q^2\sum_{n=0}^{\infty}\frac{q^{n^2+3n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}}\\
&\qquad \text{(by [11, p. 433, Eqs. (B4) and (B1)])}\\
&= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}}+\sum_{n=1}^{\infty}\frac{q^{n^2+n}(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n}}\\
&= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n}}\Big((1-q^{2n})+q^n(1-q^n)\Big)\\
&= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n}}\left((1-q^n)\left(1+q^n+q^n\right)\right)\\
&= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(q^3;q^3)_{n-1}(1+2q^n)}{(q;q)_{n-1}(q;q)_{2n}}. \qquad \square
\end{aligned}
$$

**Theorem 4.7** *Identity* (1.9) *is valid.*

**PROOF.**

$$
\begin{aligned}
&\frac{(-q^3,-q^6,q^9;q^9)_\infty(q^3,q^{15};q^{18})_\infty}{(q;q)_\infty}\\
&= \frac{f(-q^9,-q^{18})+q^3f(-1,-q^{27})}{f(-q)} \text{ (by (2.7))}\\
&= \frac{f(-q^9,-q^{18})}{f(-q)}\\
&= \sum_{n=0}^{\infty}\frac{q^{n^2+n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} \text{ (by [11, p. 433, Eq. (B3)]) .} \qquad \square
\end{aligned}
$$

**Theorem 4.8** *Identity* (1.10) *is valid.*

**PROOF.**

$$
\begin{aligned}
&\frac{(-q^4,-q^5,q^9;q^9)_\infty(q^3,q^{15};q^{18})_\infty}{(q;q)_\infty}\\
&= \frac{f(-q^6,-q^{21})+q^4f(-q^{-3},-q^{30})}{f(-q)} \text{ (by (2.7))}\\
&= \frac{f(-q^6,-q^{21})-qf(-q^3,-q^{24})}{f(-q)}\\
&= \sum_{n=0}^{\infty}\frac{q^{n^2+2n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}}-q\sum_{n=0}^{\infty}\frac{q^{n^2+3n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}}
\end{aligned}
$$

(by [11, p. 433, Eqs. (B2) and (B1)])

$$= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^3)_n(1-q^{n+1})}{(q;q)_n(q;q)_{2n+2}}$$
$$= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^3)_n}{(q;q)_n^2(q^{n+2};q)_{n+1}}. \qquad \square$$

**Theorem 4.9** *Identity* (1.11) *of Ramanujan is valid.*

**PROOF.** Set $b = e^{\pi i/3}$ and $c = 1$ in (2.10). $\square$

**Theorem 4.10** *Identity* (1.12) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.2 into Eq. (2.13) with $a = 1$ to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n}$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{r=1}^{\infty} q^{18r^2-3r}(1+q^{6r}) - \sum_{r=1}^{\infty} q^{18r^2-9r+1} - \sum_{r=0}^{\infty} q^{18r^2+9r+1}\right)$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{r=-\infty}^{\infty} q^{18r^2-3r} - q\sum_{r=-\infty}^{\infty} q^{18r^2-9r}\right)$$
$$= \frac{f(q^{15},q^{21}) - qf(q^9,q^{27})}{\psi(-q)}$$
$$= \frac{(q,q^{11},q^{12};q^{12})_\infty(q^{10},q^{14};q^{24})_\infty}{\psi(-q)} \qquad \text{(by (2.7))} . \qquad \square$$

**Theorem 4.11** *Identity* (1.13) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.1 into Eq. (2.13) with $a = 1$ to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n}$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{r=1}^{\infty} q^{18r^2-3r}(1+q^{6r}) - \sum_{r=1}^{\infty} q^{18r^2-15r+3} - \sum_{r=0}^{\infty} q^{18r^2+15r+3}\right)$$
$$= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{r=-\infty}^{\infty} q^{18r^2-3r} - q^3\sum_{r=-\infty}^{\infty} q^{18r^2-15r}\right)$$
$$= \frac{f(q^{15},q^{21}) - q^3f(q^3,q^{33})}{\psi(-q)}$$

$$= \frac{(q^3,q^9,q^{12};q^{12})_\infty (q^6,q^{18};q^{24})_\infty}{\psi(-q)} \qquad \text{(by (2.7))}\ . \qquad \Box$$

**Theorem 4.12** *Identity* (1.14) *is valid.*

**PROOF.** Set $b = e^{\pi i/3}$ and $c = q^2$ in (2.10). $\Box$

**Theorem 4.13** *Identity* (1.15) *is valid.*

**PROOF.**

$$\begin{aligned}
&\frac{(q^5,q^7,q^{12};q^{12})_\infty (q^2,q^{22};q^{24})_\infty}{\psi(-q)}\\
&= \frac{f(q^9,q^{27}) - q^5 f(q^{-3},q^{39})}{\psi(-q)}\\
&= \frac{f(q^9,q^{27}) - q^2 f(q^3,q^{33})}{\psi(-q)}\\
&= \frac{f(q^{15},q^{21}) - q^3 f(q^3,q^{33})}{q\psi(-q)} - \frac{f(q^{15},q^{21}) - q f(q^9,q^{27})}{q\psi(-q)}\\
&= q^{-1}\left(\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n(-1;q^6)_n}{(-1;q^2)_n(q^2;q^2)_{2n}} - \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n(-1;q^6)_n}{(-1;q^2)_n(q^2;q^2)_{2n}}\right)\\
&\qquad \text{(by (1.13) and (1.12))}\\
&= q^{-1}\Bigg(\sum_{n=-1}^{\infty} \frac{q^{n^2+2n+1}(-q;q^2)_{n+1}(-1;q^6)_{n+1}}{(-1;q^2)_{n+1}(q^2;q^2)_{2n+2}}\\
&\qquad\qquad - \sum_{n=-1}^{\infty} \frac{q^{n^2+4n+3}(-q;q^2)_{n+1}(-1;q^6)_{n+1}}{(-1;q^2)_{n+1}(q^2;q^2)_{2n+2}}\Bigg)\\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_{n+1}(-q^6;q^6)_n(1-q^{2n+2})}{(-q^2;q^2)_n(q^2;q^2)_{2n+2}}. \qquad \Box
\end{aligned}$$

**Theorem 4.14** *Identity* (1.16) *of Ramanujan is valid.*

**PROOF.** Set $b = e^{2\pi i/3}$ and $c = 1$ in (2.10). $\Box$

**Theorem 4.15** *Identity* (1.17) *is valid.*

**PROOF.**

$$\frac{(-q,-q^{11},q^{12};q^{12})_\infty (q^{10},q^{14};q^{24})_\infty}{\psi(-q)}$$

$$= \frac{f(-q^{15},-q^{21})+qf(-q^9,-q^{27})}{\psi(-q)} \text{ (by (2.7))}$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n-1}}+\sum_{n=0}^{\infty}\frac{q^{n^2+2n+1}(-q;q^2)_{n+1}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{2n+2}}$$
(by [11, p. 434, Eqs. (C3) and (C2)])
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n-1}}+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}}$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n}}\Big((1-q^{4n})+(1-q^{2n})\Big)$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n}}\Big((1-q^{2n})(1+q^{2n}+1)\Big)$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}(2+q^{2n})}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}}. \qquad \square$$

**Theorem 4.16** *Identity* (1.18) *is valid.*

**PROOF.**

$$\frac{(-q^3,-q^9,q^{12};q^{12})_\infty(q^6,q^{18};q^{24})_\infty}{\psi(-q)}$$
$$= \frac{f(-q^{15},-q^{21})+q^3f(-q^3,-q^{33})}{\psi(-q)} \text{ (by (2.7))}$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n-1}}+\sum_{n=0}^{\infty}\frac{q^{n^2+4n+3}(-q;q^2)_{n+1}(q^6;q^6)_n}{(q^2;q^2)_n(q^2;q^2)_{2n+2}}$$
(by [11, p. 434, Eqs. (C3) and (C1)])
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n-1}}+\sum_{n=1}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}}$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n}}\Big((1-q^{4n})+q^{2n}(1-q^{2n})\Big)$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_n(q^2;q^2)_{2n}}\Big((1-q^{2n})(1+q^{2n}+q^{2n})\Big)$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}(1+2q^{2n})}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}}. \qquad \square$$

**Theorem 4.17** *Identity* (1.19) *of Slater is valid.*

**PROOF.** Set $b=e^{2\pi i/3}$ and $c=q^2$ in (2.10). $\square$

**Theorem 4.18** *Identity* (1.20) *of Slater is valid.*

**PROOF.**

$$
\begin{aligned}
&\frac{(-q^5,-q^7,q^{12};q^{12})_\infty(q^2,q^{22};q^{24})_\infty}{\psi(-q)}\\
&=\frac{f(-q^9,-q^{27})+q^5f(-q^{-3},-q^{39})}{\psi(-q)}\\
&=\frac{f(-q^9,-q^{27})-q^2f(-q^3,-q^{33})}{\psi(-q)}\\
&=\frac{f(-q^{15},-q^{21})+qf(-q^9,-q^{27})}{q\psi(-q)}-\frac{f(-q^{15},-q^{21})+q^3f(-q^3,-q^{33})}{q\psi(-q)}\\
&=q^{-1}\Bigg(1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}(2+q^{2n})}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}}-1\\
&\qquad\qquad-\sum_{n=1}^{\infty}\frac{q^{n^2}(-q;q^2)_n(q^6;q^6)_{n-1}(1+2q^{2n})}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}}\Bigg)\\
&\qquad\text{(by (1.17) and (1.18))}\\
&=\sum_{n=1}^{\infty}\frac{q^{n^2-1}(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}}\big((2+q^{2n})-(1+2q^{2n})\big)\\
&=\sum_{n=1}^{\infty}\frac{q^{n^2-1}(-q;q^2)_n(q^6;q^6)_{n-1}(1-q^{2n})}{(q^2;q^2)_{n-1}(q^2;q^2)_{2n}}\\
&=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_{n+1}(q^6;q^6)_n(1-q^{2n+2})}{(q^2;q^2)_n(q^2;q^2)_{2n+2}}. \qquad \square
\end{aligned}
$$

**Remark 4.19** *Identity* (1.21) *follows from* (1.23) *and* (1.25)*, and is therefore recorded after* (1.25).

**Theorem 4.20** *Identity* (1.22) *is valid.*

**PROOF.** Set $a=e^{\pi i/3}$, $b=e^{-\pi i/3}$ in (2.9). $\square$

**Theorem 4.21** *Identity* (1.23) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.4 into (2.14) to get that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n}\\
&=\frac{1}{\varphi(-q^2)}\left(\sum_{r\geqq 0}q^{18r^2+3r}+\sum_{r\geqq 1}q^{18r^2-3r}-\sum_{r\geqq 0}q^{18r^2+15r+3}(1+q^{6r+3})\right)\\
&=\frac{1}{\varphi(-q^2)}\left(\sum_{r\in\mathbb{Z}}q^{18r^2+3r}-q^3\sum_{r\in\mathbb{Z}}q^{18r^2+15r}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-q^{15},-q^{21},q^{36};q^{36})_\infty - q^3(-q^3,-q^{33},q^{36};q^{36})_\infty}{\varphi(-q^2)} \qquad (4.1)\\
&= \frac{(q^3,q^9,q^{12};q^{12})_\infty (q^6,q^{18};q^{24})_\infty}{\varphi(-q^2)},
\end{aligned}
$$

where the last equality follows from (2.7). □

**Theorem 4.22** *Identity* (1.24) *is valid.*

**PROOF.** Set $a = e^{\pi i/3}q^2$, $b = e^{-\pi i/3}q^2$ in (2.9). □

**Theorem 4.23** *Identity* (1.25) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.5 into (2.14) to get that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n(n+3)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n}\\
&= \frac{1}{\varphi(-q^2)}\left(\sum_{r\geqq 0} q^{18r^2+9r} + \sum_{r\geqq 1} q^{18r^2-9r} - \sum_{r\geqq 0} q^{18r^2+9r+2}(q^{12r+3}+q^{6r})\right)\\
&= \frac{1}{\varphi(-q^2)}\left(\sum_{r\in\mathbb{Z}} q^{18r^2+9r} - q^2\sum_{r\in\mathbb{Z}} q^{18r^2+15r}\right)\\
&= \frac{(-q^9,-q^{27},q^{36};q^{36})_\infty - q^2(-q^3,-q^{33},q^{36};q^{36})_\infty}{\varphi(-q^2)} \qquad (4.2)\\
&= \frac{(-q^9,-q^{27},q^{36};q^{36})_\infty - q^5(-q^{39},-q^{-3},q^{36};q^{36})_\infty}{\varphi(-q^2)}\\
&= \frac{(q^7,q^5,q^{12};q^{12})_\infty (q^2,q^{22};q^{24})_\infty}{\varphi(-q^2)},
\end{aligned}
$$

where the last equality follows, as above, from (2.7). □

**Theorem 4.24** *Identity* (1.21) *is valid.*

**PROOF.**

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q;q)_{2n}(-q;q)_{2n+1}(-q;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(1-q^{2n+1})(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n}\\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n} - q\sum_{n=0}^{\infty} \frac{q^{n(n+3)}(-q^2;q^2)_n(-q^3;q^6)_n}{(q^2;q^2)_{2n+1}(-q;q^2)_n}\\
&= \frac{(-q^{15},-q^{21},q^{36};q^{36})_\infty - q^3(-q^3,-q^{33},q^{36};q^{36})_\infty}{\varphi(-q^2)}
\end{aligned}
$$

$$- \frac{q(-q^9,-q^{27},q^{36};q^{36})_\infty - q^3(-q^3,-q^{33},q^{36};q^{36})_\infty}{\varphi(-q^2)}$$

(by (4.2) and (4.1))

$$= \frac{(-q^{15},-q^{21},q^{36};q^{36})_\infty - q(-q^9,-q^{27},q^{36};q^{36})_\infty}{\varphi(-q^2)}$$

$$= \frac{(q,q^{11},q^{12};q^{12})_\infty(q^{10},q^{14};q^{24})_\infty}{\varphi(-q^2)},$$

where the last equality follows, as above, from (2.7). □

**Remark 4.25** *Identity* (1.26) *follows from* (1.28) *and* (1.30) *and is therefore recorded after* (1.30).

**Theorem 4.26** *Identity* (1.27) *is valid.*

**PROOF.** Set $a = e^{2\pi i/3}$, $b = e^{-2\pi i/3}$ in (2.9). □

**Theorem 4.27** *Identity* (1.28) *of Slater is valid.*

**PROOF.** Insert Slater's Bailey pair J4 [34, p. 149] into (2.14). □

**Theorem 4.28** *Identity* (1.29) *of Dyson is valid.*

**PROOF.** Set $a = e^{2\pi i/3}q^2$, $b = e^{-2\pi i/3}q^2$ in (2.9). □

**Theorem 4.29** *Identity* (1.30) *is valid.*

**PROOF.** Insert Slater's Bailey pair J5 [34, p. 149] into (2.14). □

**Theorem 4.30** *Identity* (1.26) *is valid.*

**PROOF.**

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n(q^3;q^6)_n(1+q^{2n+1})}{(q^2;q^2)_{2n+1}(q;q^2)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^2;q^2)_n(q^3;q^6)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n} + q\sum_{n=0}^{\infty} \frac{q^{n(n+3)}(-q^2;q^2)_n(q^3;q^6)_n}{(q^2;q^2)_{2n+1}(q;q^2)_n}$$

$$= \frac{(-q^3,-q^9,q^{12};q^{12})_\infty(q^6,q^{18};q^{24})_\infty + q(-q^5,-q^7,q^{12};q^{12})_\infty(q^2,q^{22};q^{24})_\infty}{\varphi(-q^2)}$$

(by (1.28) and (1.30))

$$
\begin{aligned}
&= \frac{f(-q^{15},-q^{21}) + q^3 f(-q^3,-q^{33}) + qf(q^9,q^{27}) + q^6 f(-q^{-3},-q^{-39})}{\varphi(-q^2)} \text{ (by (2.7))}\\
&= \frac{f(-q^{15},-q^{21}) + q^3 f(-q^3,-q^{33}) + qf(q^9,q^{27}) - q^3 f(-q^3,-q^{-33})}{\varphi(-q^2)}\\
&= \frac{f(-q^{15},-q^{21}) + qf(q^9,q^{27})}{\varphi(-q^2)}\\
&= \frac{(-q,-q^{11},q^{12};q^{12})_\infty (q^{10},q^{14};q^{24})_\infty}{\varphi(-q^2)} \text{ (by (2.7))} \qquad \square
\end{aligned}
$$

## 5 False theta series identities

Rogers introduced the term "false theta series" and included a number of related identities in his 1917 paper [30]. Ramanujan presented a number of identities involving false theta series in his lost notebook [8, p. 256–259, §11.5].

Recalling that Ramanujan defines the theta function as

$$
\begin{aligned}
f(a,b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}\\
&= \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} + \sum_{n=1}^{\infty} a^{n(n-1)/2} b^{n(n+1)/2}\\
&= 1 + a + b + a^3 b + ab^3 + a^6 b^3 + a^3 b^6 + a^{10} b^6 + a^6 b^{10} + \dots,
\end{aligned}
$$

let us define the corresponding *false theta function* as

$$
\begin{aligned}
\Psi(a,b) &:= \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} - \sum_{n=1}^{\infty} a^{n(n-1)/2} b^{n(n+1)/2}\\
&= \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} (1 - b^{2n+1})\\
&= 1 + a - b + a^3 b - ab^3 + a^6 b^3 - a^3 b^6 + a^{10} b^6 - a^6 b^{10} + \dots.
\end{aligned}
$$

In practice, $a$ and $b$ are always taken to be $\pm q^h$ for some integer or half-integer $h$.

We now prove the false theta series identities (1.31)–(1.40).

**Theorem 5.1** *Identity* (1.32) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.6 into (2.15) and the result follows after some simple manipulations. $\square$

**Theorem 5.2** *Identity* (1.33) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.4 into (2.15). $\square$

**Theorem 5.3** *Identity* (1.34) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.3 into (2.15). $\square$

**Theorem 5.4** *Identity* (1.35) *is valid.*

**PROOF.** Insert the Bailey pair from Lemma 3.5 into (2.15). $\square$

**Theorem 5.5** *Identity* (1.31) *is valid.*

**PROOF.** Identity (1.31) is (1.33)$-q\times$(1.35). $\square$

**Theorem 5.6** *Identity* (1.37) *of Dyson is valid.* $\square$

**PROOF.** Insert the Bailey pair from Lemma 3.7 into (2.15) to get, after multiplying both sides by $1-q^2$, that

$$\begin{aligned}1+(1-q^2)\sum_{n\geqq 1}(-1)^n q^{n(n+1)}\frac{(q^6;q^6)_{n-1}(1-q^{2n})}{(q^2;q^2)_{2n}}&\\ =\sum_{r\geqq 0}(q^{18r^2}-q^{18r^2+12r+2})+\sum_{r\geqq 1}(q^{18r^2-12r+2}-q^{18r^2})&\\ +\sum_{r\geqq 0}(q^{18r^2+12r+4}-q^{18r^2+24r+10}).&\end{aligned}$$

The result follows after noting that the two sums on the right over terms of the form $q^{18r^2}$ cancel to leave 1, subtracting 1 from both sides, re-indexing the left side by replacing $n$ with $n+1$, re-indexing what remains of the middle sum on the right side by replacing $r$ with $r+1$, rearranging the right side and finally canceling a factor of $-q^2(1-q^2)$ on both sides. $\square$

**Remark 5.7** *Although Dyson contributed Identity* (1.37) *(among others) to Bailey's paper [11, p. 434, Eq. (E1)], Bailey remarks [11, p. 434], "Mr. Dyson tells me that the method of proof is a standard type of argument due to Rogers," but elaborates no further. Thus, it would appear that the first published proof of* (1.37) *is given here. A similar remark would apply to* (1.39)*, which Dyson also contributed to Bailey's paper [11, p. 434, Eq. (E2)], were it not for the fact that Ramanujan had included* (1.39) *in his lost notebook, and thus a proof appears in Andrews and Berndt's volume [9, Entry 5.4.2].*

**Theorem 5.8** *Identity* (1.38) *is valid.*

**PROOF.** Insert Slater's Bailey pair J4 [34, p. 149] into (2.15). □

**Theorem 5.9** *Identity* (1.39) *of Ramanujan and Dyson is valid.*

**PROOF.** See [9, Entry 5.4.2].

**Theorem 5.10** *Identity* (1.40) *is valid.*

**PROOF.** Insert Slater's Bailey pair J5 [34, p. 149] into (2.15). □

**Theorem 5.11** *Identity* (1.36) *is valid.*

**PROOF.** Identity (1.36) is (1.38)+$q\times$(1.40). □

## 6 Connections with Lie algebras

Let $\mathfrak{g}$ be the affine Kac-Moody Lie algebra $A_1^{(1)}$ or $A_2^{(2)}$. Let $h_0, h_1$ be the usual basis of a maximal toral subalgebra $T$ of $\mathfrak{g}$. Let $d$ denote the "degree derivation" of $\mathfrak{g}$ and $\tilde{T} := T \oplus \mathbb{C}d$. For all dominant integral $\lambda \in \tilde{T}^*$, there is an essentially unique irreducible, integrable, highest weight module $L(\lambda)$, assuming without loss of generality that $\lambda(d) = 0$. Now $\lambda = s_0\Lambda_0 + s_1\Lambda_1$ where $\Lambda_0$ and $\Lambda_1$ are the fundamental weights, given by $\Lambda_i(h_j) = \delta_{ij}$ and $\Lambda_i(d) = 0$; here $s_0$ and $s_1$ are nonnegative integers. For $A_1^{(1)}$, the canonical central element is $c = h_0 + h_1$, while for $A_2^{(2)}$, the canonical central element is $c = h_0 + 2h_1$. The quantity $\lambda(c)$ (which equals $s_0 + s_1$ for $A_1^{(1)}$ and which equals $s_0 + 2s_1$ for $A_2^{(2)}$) is called the *level* of $L(\lambda)$. (cf.[21], [23].)

Additionally (see [23]), there is an infinite product $F_{\mathfrak{g}}$ associated with $\mathfrak{g}$, often light-heartedly called the "fudge factor," which needs to be divided out of the the principally specialized character $\chi(L(\lambda)) = \chi(s_0\Lambda_0 + s_1\Lambda_1)$, in order to obtain the quantities of interest here. For $\mathfrak{g} = A_1^{(1)}$, the fudge factor is given by $F_{\mathfrak{g}} = (q;q^2)_\infty^{-1}$, while for $\mathfrak{g} = A_2^{(2)}$, it is given by $F_{\mathfrak{g}} = [(q;q^6)_\infty (q^5;q^6)_\infty]^{-1}$.

Now $\mathfrak{g}$ has a certain infinite-dimensional Heisenberg subalgebra known as the "principal Heisenberg vacuum subalgebra" $\mathfrak{s}$ (see [24] for the construction of $A_1^{(1)}$ and [22] for that of $A_2^{(2)}$). As shown in [25], the principal character $\chi(\Omega(s_0\Lambda_0 + s_1\Lambda_1))$, where $\Omega(\lambda)$ is the vacuum space for $\mathfrak{s}$ in $L(\lambda)$, is

$$\chi(\Omega(s_0\Lambda_0 + s_1\Lambda_1)) = \frac{\chi(L(s_0\Lambda_0 + s_1\Lambda_1))}{F_{\mathfrak{g}}}, \tag{6.1}$$

where $\chi(L(\lambda))$ is the principally specialized character of $L(\lambda)$.

By [23] applied to (6.1) in the case of $A_1^{(1)}$, for standard modules of odd level $2k+1$,

$$\chi(\Omega((2k-i+2)\Lambda_0 + (i-1)\Lambda_1))$$

is given by Andrews' analytic generalization of the Rogers-Ramanujan identities [3]:

$$\sum_{n_1,n_2,\dots,n_k \geqq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_k^2+N_i+N_{i+1}+\cdots+N_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_k}} = \frac{(q^i, q^{2k+3-i}, q^{2k+3}; q^{2k+3})_\infty}{(q;q)_\infty}, \tag{6.2}$$

where $1 \leqq i \leqq k+1$ and $N_j := n_j + n_{j+1} + \cdots + n_k$. The combinatorial counterpart to (6.2) is Gordon's partition theoretic generalization of the Rogers-Ramanujan identities [20]; this generalization was explained vertex-operator theoretically in [26] and [27].

In addition, for the $A_1^{(1)}$ standard modules of even level $2k$,

$$\chi(\Omega((2k-i+1)\Lambda_0 + (i-1)\Lambda_1))$$

is given by Bressoud's analytic identity [15, p. 15, Eq. (3.4)]

$$\sum_{n_1,n_2,\dots,n_k \geqq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_k^2+N_i+N_{i+1}+\cdots+N_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}(q^2;q^2)_{n_k}} = \frac{(q^i, q^{2k+2-i}, q^{2k+2}; q^{2k+2})_\infty}{(q;q)_\infty}, \tag{6.3}$$

where $1 \leqq i \leqq k+1$, and its partition theoretic counterpart [14, p. 64, Theorem, $j=0$ case]; likewise, this generalization was explained vertex-operator theoretically in [26] and [27].

Notice that the infinite products associated with level $\ell$ standard modules for $A_1^{(1)}$ in (6.2) and (6.3) are instances of the Jacobi triple product identity for modulus $\ell+2$ divided by $(q;q)_\infty$.

Probably the most efficient way of deriving (6.2) is via the Bailey lattice [1], which is an extension of the Bailey chain concept ([4]; cf. [5, §3.5, pp. 27ff]) built upon the "unit Bailey pair"

$$\beta_n(1,q) = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n > 0 \end{cases}$$

$$\alpha_n(1,q) = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(n-1)/2}(1+q^n) & \text{if } n>0. \end{cases}$$

Similarly, (6.3) follows from a Bailey lattice built upon the Bailey pair

$$\beta_n(1,q) = \frac{1}{(q^2;q^2)_n},$$
$$\alpha_n(1,q) = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n 2q^{n^2} & \text{if } n>0. \end{cases}$$

Thus the standard modules of $A_1^{(1)}$ may be compactly "explained" via two interlaced instances of the Bailey lattice.

In contrast, the standard modules of $A_2^{(2)}$ are not as well understood, and a uniform $q$-series and partition correspondence analogous to what is known for $A_1^{(1)}$ has thus far remained elusive.

As with $A_1^{(1)}$, there are $1+\lfloor\frac{\ell}{2}\rfloor$ inequivalent level $\ell$ standard modules associated with the Lie algebra $A_2^{(2)}$, but the analogous quantity for the level $\ell$ standard modules
$$\chi(\Omega((\ell-2i+2)\Lambda_0+(i-1)\Lambda_1))$$
is given by instances of the quintuple product identity (rather than the triple product identity) divided by $(q;q)_\infty$:

$$\frac{(q^i,q^{\ell+3-i},q^{\ell+3};q^{\ell+3})_\infty (q^{\ell+3-2i},q^{2\ell+2i+3};q^{2\ell+6})_\infty}{(q;q)_\infty}, \tag{6.4}$$
where $1 \leqq i \leqq 1+\lfloor\frac{\ell}{2}\rfloor$; see [23].

It seems quite plausible that in the case of $A_2^{(2)}$, the analog of the Andrews-Gordon-Bressoud identities would involve the interlacing of six Bailey lattices in contrast to the two that were necessary for $A_1^{(1)}$. To see this, consider the following set of Andrews-Gordon-Bressoud type identities where the product sides involve instances of the quintuple product identity rather than the triple product identity:

$$\sum_{n_1,n_2,\ldots,n_k\geqq 0} \frac{q^{N_1(N_1+1)/2+N_2(N_2+1)+N_3(N_3+1)+\cdots+N_k(N_k+1)+N_k^2}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}(q;q)_{2n_k+1}(-q^{N_1+1};q)_\infty}$$
$$= \frac{(q^k,q^{5k-1},q^{6k-1};q^{6k-1})_\infty(q^{4k-1},q^{8k-1};q^{12k-2})_\infty}{(q;q)_\infty} \tag{6.5}$$

$$\sum_{n_1,n_2,\dots,n_{k+1}\geqq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_k^2}\left(\frac{n_k-n_{k+1}+1}{3}\right)}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k+1}}(q;q)_{2n_k-n_{k+1}}} = \frac{(q^k,q^{5k},q^{6k};q^{6k})_\infty(q^{4k},q^{8k};q^{12k})_\infty}{(q;q)_\infty} \tag{6.6}$$

$$\sum_{n_1,n_2,\dots,n_k\geqq 0} \frac{q^{N_1(N_1+1)/2+N_2(N_2+1)+N_3(N_3+1)+\cdots+N_k(N_k+1)}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}(q;q)_{2n_k+1}(-q^{N_1+1};q)_\infty} = \frac{(q^{2k},q^{4k+1},q^{6k+1};q^{6k+1})_\infty(q^{2k+1},q^{10k+1};q^{12k+2})_\infty}{(q;q)_\infty} \tag{6.7}$$

$$\sum_{n_1,n_2,\dots,n_k\geqq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+2N_k^2}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}(q;q)_{2n_k}} = \frac{(q^k,q^{5k+2},q^{6k+2};q^{6k+2})_\infty(q^{4k+2},q^{8k+2};q^{12k+4})_\infty}{(q;q)_\infty} \tag{6.8}$$

$$\sum_{n_1,n_2,\dots,n_k\geqq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_k^2}(-1;q^3)_{n_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}(q;q)_{2n_k}(-1;q)_{n_k}} = \frac{(q^{k+1},q^{5k+2},q^{6k+3};q^{6k+3})_\infty(q^{4k+1},q^{8k+5};q^{12k+6})_\infty}{(q;q)_\infty} \tag{6.9}$$

$$\sum_{n_1,n_2,\dots,n_k\geqq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_k^2}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}(q;q)_{2n_k}} = \frac{(q^{k+1},q^{5k+3},q^{6k+4};q^{6k+4})_\infty(q^{4k+2},q^{8k+6};q^{12k+8})_\infty}{(q;q)_\infty}, \tag{6.10}$$

where $\left(\frac{n}{p}\right)$ in (6.6) is the Legendre symbol. We note that (6.6) first appeared in [32, p. 400, Eq. (1.7)] and that (6.10) is due to Andrews [4, p. 269, Eq. (1.8)]. While (6.5), (6.7), and (6.8) probably have not appeared explicitly in the literature, they each follow from building a Bailey chain on a known Bailey pair and may be regarded as nothing more than a standard exercise in light of Andrews' discovery of the Bailey chain ([4]; cf. [5, §3.5]). Indeed the $k = 1$ cases of (6.5), (6.7), (6.8), and (6.10) are all due to Rogers and appear in Slater's list [34] as Eqs. (62), (80), (83), and (98) respectively. On the other hand, (6.9) is new since it arises from inserting a new Bailey pair, namely the one from Lemma 3.1 in this paper, into the Bailey chain mechanism. Notice that as $k$ runs through the positive integers in the numerators of the right hand sides of (6.5)–(6.10), we obtain instances of the quintuple product identity for all moduli represented in (6.4) (except for the trivial level 1 case where the relevant identity reduces to "$1 = 1$"). It is because of the preceding observations that we conjecture that $A_2^{(2)}$ may be "explained" by six interlaced Bailey lattices.

We now turn our attention to combinatorial considerations in the context of $A_2^{(2)}$. In his 1988 Ph.D. thesis S. Capparelli [16] conjectured two beautiful partition identities resulting from his analysis of the two inequivalent level 3 standard modules of $A_2^{(2)}$, using the theory in [26] and [27]. Capparelli's conjectures were first proved by Andrews [6] using combinatorial methods. Later, Lie algebraic proofs were found by Tamba and Xie [35] and Capparelli himself [17]. More recently, Capparelli [18] related the principal characters of the vacuum spaces for the standard modules of $A_2^{(2)}$ for levels 5 and 7 to some known $q$-series and partition identities. In the same way, our identities (1.3)–(1.6) appear to correspond to the standard modules for level 6.

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