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Notes on Stewart's Calculus: Concepts & Contexts, 4th edition
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§2.1—The Tangent and Velocity Problems

Example 1. The distance in feet that an object falls in t seconds under the influence of gravity is given by $y = 16t^2$. If a ball is dropped from the top of a tall building, calculate its average speed

(a) between $t = 0$ and $t = 3$

(b) between $t = 2$ and $t = 3$

(c) between $t = 2.5$ and $t = 3$

(d) between $t = 2.9$ and $t = 3$

(e) between $t = 2.99$ and $t = 3$

What appears to be the instantaneous speed of the ball at $t = 3$?

Definition. The **average rate of change** of $y = f(x)$ over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

It is the slope of the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

By allowing the point x_2 to move closer and closer to x_1, we obtain the **tangent line** to the graph of $y = f(x)$ at the point $x = x_1$.

Example 2. Find the slope of the secant line on the graph of $f(x) = e^x$ for each of the following intervals.

(a) $[0, 1]$

(b) $[0, 0.5]$

(c) $[0, 0.1]$

(d) $[0, 0.01]$

What appears to be the slope of the tangent line to the graph at $x = 0$? What is the equation of this tangent line?

§2.2—The Limit of a Function

To make sense of instantaneous rates of change, we need the concept of a limit.

Definition. If we can make $f(x)$ as close as we like to L by taking x sufficiently close to a, then we say that f approaches the **limit** L as x approaches a, and we write

$$\lim_{x \to a} f(x) = L.$$

Important points:

1. The limit may or may not exist.
2. We must consider values on both sides of a.
3. The value of f at $x = a$ is irrelevant.

Example 1. Use numerical data to guess the values of the following limits.

(a) $\displaystyle\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

(b) $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$

(c) $\displaystyle\lim_{x \to 3} \frac{1}{(x - 3)^2}$

One-sided limits

$\lim_{x\to a^+} f(x) = L$ means $f(x)$ approaches L as x approaches a from the right

$\lim_{x\to a^-} f(x) = L$ means $f(x)$ approaches L as x approaches a from the left

Fact. We have $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

Example 2. Compute each of the following limits for the function graphed below.

(a) $\lim_{x\to 2^+} f(x)$ (b) $\lim_{x\to 2^-} f(x)$ (c) $\lim_{x\to 2} f(x)$

(d) $\lim_{x\to 5^+} f(x)$ (e) $\lim_{x\to 5^-} f(x)$ (f) $\lim_{x\to 5} f(x)$

Example 3. Evaluate each of the following.

(a) $\lim_{x\to 0^+} \frac{2x}{|x|}$

(b) $\lim_{x\to 0^-} \frac{2x}{|x|}$

(c) $\lim_{x\to 0} \frac{2x}{|x|}$

§2.3—Calculating Limits Using the Limit Laws

Limit Laws. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

$$\lim_{x\to a}(f(x) \pm g(x)) = L \pm M \qquad \lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M$$

$$\lim_{x\to a}(k \cdot f(x)) = k \cdot L \qquad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ provided that } M \neq 0$$

$$\lim_{x\to a}(f(x))^{r/s} = L^{r/s} \text{ provided that } L^{r/s} \text{ is a real number.}$$

Example 1. Evaluate $\lim_{x\to 2} \frac{x^4 + x^2 - 1}{x^2 + 5}$.

Since $\lim_{x\to a} k = k$ and $\lim_{x\to a} x = a$, limits of polynomials and rational functions can be found by direct substitution, *provided the limit of the denominator is not zero.*

What if the denominator does approach zero? If the numerator also approaches zero, we may be able to find the limit by canceling a common factor.

Example 2. Evaluate $\lim_{x\to 3} \frac{x^2 - 9}{x^2 - 2x - 3}$.

Example 3. Evaluate $\lim_{x \to 25} \frac{25 - x}{5 - \sqrt{x}}$.

Example 4. Evaluate $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$.

Example 5. Evaluate $\lim_{x \to 4} \frac{\sqrt{2x + 1} - 3}{x - 4}$.

Example 6. Evaluate the limit $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ when $f(x) = 1/x$ and $x = 2$.

Note: This is the instantaneous rate of change of f at $x = 2$.

The Squeeze Theorem. Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing a, except possibly at a itself. If $\lim_{x\to a} g(x) = L$ and $\lim_{x\to a} h(x) = L$ then we can conclude that $\lim_{x\to a} f(x) = L$.

Example 7. Calculate $\lim_{x\to 0} x^2 \sin(1/x)$.

§2.4—Continuity

We say that f is **continuous** at an interior point a if $\lim_{x \to a} f(x) = f(a)$.

This implicitly requires checking three things:

(i) $f(a)$ exists (ii) $\lim_{x \to a} f(x)$ exists (iii) the numbers in (i) and (ii) are equal.

If a is an endpoint of the domain, we use the appropriate one-sided limit instead.

A point where f is not continuous is called a **discontinuity**.

Example 1. At what points does the function graphed below fail to be continuous?

Useful facts:

1. Sums, differences, products, quotients, powers, roots, and compositions of continuous functions are continuous at all points of their domains.

2. Polynomials, rational functions, root functions, trigonometric functions, exponentials, and logarithms are continuous at all points of their domains.

Example 2. For what values of x is the function $f(x) = \dfrac{2 + \sin(x^2)}{\sqrt{x^2 - 4} - 1}$ continuous?

Example 3. For what values of c is $f(x) = \begin{cases} cx + 1 & \text{if } x \leq 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases}$ continuous at $x = 3$?

If f is discontinuous at a but $\lim_{x \to a} f(x)$ exists, then we say the discontinuity is **removable**. If we just change the value of $f(a)$ to match the limit, we get a continuous function.

Example 4. Identify the discontinuities of each function, and state whether they are removable.

(a) $f(x) = \dfrac{x^2 - 4}{x - 2}$

(b) $f(x) = \dfrac{x^2 - 4}{|x - 2|}$

The Intermediate Value Theorem. If f is continuous on $[a, b]$, then f takes on every value between $f(a)$ and $f(b)$. In other words, if N is between $f(a)$ and $f(b)$, then there is some c in $[a, b]$ for which $f(c) = N$.

Example 5 (Root-finding). Show that the equation $x^3 + x + 1 = 0$ has at least one solution in the interval $[-1, 0]$.

§2.5—Limits Involving Infinity

We write $\lim_{x \to a} f(x) = \infty$ if the values of $f(x)$ become arbitrarily large and positive as x approaches a. Similarly, we write $\lim_{x \to a} f(x) = -\infty$ if the values of $f(x)$ become arbitrarily large and negative as x approaches a. Similar definitions apply to one-sided infinite limits.

The line $x = a$ is called a **vertical asymptote** for the graph of $y = f(x)$ if

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

Example 1. Evaluate each of the following limits.

(a) $\lim_{x \to 0^-} \dfrac{2}{3x}$

(b) $\lim_{x \to 2} \dfrac{1}{(x-2)^4}$

(c) $\lim_{x \to 5^+} \dfrac{x+3}{x-5}$

(d) $\lim_{x \to \pi^-} \cot x$

Example 2. If $f(x) = \dfrac{x}{x^2 - 1}$, evaluate

(a) $\lim_{x \to 1^+} f(x)$

(b) $\lim_{x \to 1^-} f(x)$

(c) $\lim_{x \to -1^+} f(x)$

(d) $\lim_{x \to -1^-} f(x)$

Limits at Infinity

We say that $\lim_{x\to\infty} f(x) = L$ if $f(x)$ can be made as close as we like to L by taking x sufficiently large. Similarly, we say that $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ can be made as close as we like to L by taking $-x$ sufficiently large (that is, $|x|$ sufficiently large and $x < 0$).

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is called a **horizontal asymptote** for the graph of $y = f(x)$.

All the limit laws from §2.3 hold for these types of limits as well.

Example 3. Estimate the value of $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.

Example 4. Evaluate the following limits.

(a) $\lim_{x\to\infty} \left(4 + \frac{5}{x^2} + e^{-x}\right)$

(b) $\lim_{x\to\infty} \frac{\sin x}{x}$

(c) $\lim_{x\to-\infty} e^x \cos 2x$

Example 5. Evaluate each limit by dividing through by the largest term in the denominator.

(a) $\lim_{x\to\infty} \dfrac{8x^3 + 5x + 1}{2x^3 + 4}$

(b) $\lim_{x\to-\infty} \dfrac{3x^2 - 10}{x^5 + 3x + 1}$

Example 6. Sketch the graph of the rational function $f(x) = \dfrac{x+2}{x+1}$.

§2.6—Derivatives and Rates of Change

The slope of the **secant line** connecting the points $P(a, f(a))$ and $Q(a+h, f(a+h))$ on the graph of f is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

This is the **average rate of change** of f over the interval $[a, a+h]$.

The slope of the **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. This is the **instantaneous rate of change** of f at $x = a$ and is also called the **derivative** of f at $x = a$. By setting $x = a + h$, we can alternatively write

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

which is sometimes easier to work with.

Example 1. The distance in feet that an object falls in t seconds under the influence of gravity is given by $y = 16t^2$. If a ball is dropped from the top of a tall building, calculate its instantaneous speed after 3 seconds. [Compare with Example 1 in §2.1 notes.]

Example 2. Find the equation of the tangent line to the curve $y = \sqrt{x+1}$ at the point $(3, 2)$.

Example 3. Find the equation of the tangent line to the curve $y = 1/x^2$ at the point $(-1, 1)$.

§2.7—The Derivative as a Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The process of calculating a derivative is called **differentiation**.

We sometimes write $\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$ instead of $f'(x)$.

Example 1. Use the above definition to find the derivative of the following functions.

(a) $f(x) = x^2$

(b) $g(t) = \frac{1}{t}$

Example 2. Given the graph of $f(x)$ below, sketch the graph of $f'(x)$.

A function $f(x)$ is **differentiable** at $x = c$ if $f'(c)$ exists. There are several ways a function can fail to be differentiable:

1. Corner
2. Cusp
3. Vertical tangent
4. Discontinuity

Theorem. Differentiability implies continuity. In other words, if f has a derivative at $x = c$ then f is continuous at $x = c$.

The converse of this theorem is false! Continuity does NOT imply differentiability—see the corner, cusp, and vertical tangent examples above.

Example 3. At what points does the function graphed below fail to be differentiable?

Higher Derivatives:

$$f''(x) = \frac{d}{dx}[f'(x)] = \frac{d^2y}{dx^2} \qquad \text{(2nd derivative)}$$

$$f'''(x) = \frac{d}{dx}[f''(x)] = \frac{d^3y}{dx^3} \qquad \text{(3rd derivative)}$$

$$f^{(4)}(x) = \frac{d}{dx}[f'''(x)] = \frac{d^4y}{dx^4} \qquad \text{(4th derivative)}$$

§3.1—Derivatives of Polynomials and Exponential Functions

Some Basic Rules:

1. Derivative of a Constant Function: $\dfrac{d}{dx}(c) = 0$
2. Power Rule: $\dfrac{d}{dx}(x^n) = nx^{n-1}$ when n is a constant.
3. Derivative of the Natural Exponential Function: $\dfrac{d}{dx}(e^x) = e^x$
4. Constant Multiples: $\dfrac{d}{dx}[cf(x)] = cf'(x)$
5. Sums and Differences: $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Example 1. Find the derivative of each of the following functions.

(a) $f(x) = 3x^5 - 4x^3 + 2x + 6$

(b) $g(x) = \dfrac{2}{x} - \dfrac{3}{x^4}$

(c) $h(x) = 4e^x + 10\sqrt{x} + e^2$

Example 2. Find the equation of the tangent line to the graph of $f(x) = x^4 + 3x + 1$ at the point $(1, 5)$.

§3.2—The Product and Quotient Rules

Differentiating products and quotients is not quite as simple as differentiating sums and differences. For example, consider writing x^5 as the product $x^3 \cdot x^2$. The product of the derivatives of the two factors in the second expression is $3x^2 \cdot 2x = 6x^3$, but we know that the derivative of this product is really $5x^4$. This shows that the derivative of a product is NOT equal to the product of the derivatives. Instead we have:

The Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Example 1. Find the derivative of each of the following functions.

(a) $h(x) = (x^2 + 3x + 1)e^x$

(b) $z(t) = (3\sqrt{t} + 2)(4 - t^{-5})$

Example 2. Compute $f''(1)$ for the function $f(x) = x^2e^x$.

Likewise, simple examples show that the derivative of a quotient is NOT equal to the quotient of the derivatives. The correct result is as follows:

The Quotient Rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example 3. Find the derivative of each of the following functions.

(a) $h(x) = \dfrac{x}{x^5 + 3}$

(b) $z(t) = \dfrac{t^2 + 3e^t}{2e^t - t}$

(c) $F(y) = \dfrac{y - \sqrt{y}}{y^{1/3}}$

Example 4. Find the equation of the tangent line to the curve $y = \dfrac{e^x}{x}$ at the point $(1, e)$.

§3.3—Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \qquad\qquad \frac{d}{dx}(\cot x) =$$

$$\frac{d}{dx}(\cos x) = \qquad\qquad \frac{d}{dx}(\sec x) =$$

$$\frac{d}{dx}(\tan x) = \qquad\qquad \frac{d}{dx}(\csc x) =$$

To find the derivatives of $\sin x$ and $\cos x$ we use the trigonometric identities

$$\sin(x+h) = \sin x \cos h + \cos x \sin h \quad \text{and} \quad \cos(x+h) = \cos x \cos h - \sin x \sin h$$

along with the limits $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.

Example 1. Find the derivatives of the following functions.

(a) $f(x) = x^2 \sin x + 2 \cos x$

(b) $g(t) = \dfrac{e^t}{t - \sin t}$

Example 2. Find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$.

Example 3. Find the derivatives of the following functions.

(a) $f(x) = e^x \sec x + 2\tan x$

(b) $r(\theta) = \dfrac{1 + \cot\theta}{\theta^3 - 4\csc\theta}$

Example 4. Find the equation of the tangent line to $y = 2\cos x$ at the point $(\pi/3, 1)$.

§3.4—The Chain Rule

How do we differentiate compositions of functions like e^{2x}, $\cos(x^2)$, $\sqrt{x^3+1}$, or $\sin^4 x$?

Suppose that $y = f(u)$ and $u = g(x)$, so that $y = f(g(x))$. It is helpful to think of f as the "outer" function and g as the "inner" function.

If we have $\frac{du}{dx} = 2$ and $\frac{dy}{du} = 3$, then a 1 unit change in x gives approximately 2 units change in u, which then gives approximately 6 units change in y. This heuristic argument suggests that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u)g'(x)$, which is known as the Chain Rule.

The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

Example 1. Find the derivatives of the following functions.

(a) $h(x) = e^{2x}$

(b) $h(x) = \cos(x^2)$

(c) $h(x) = \sqrt{x^3+1}$

(d) $h(x) = \sin^4 x$

Example 2. Find formulas for the velocity and acceleration of a particle whose position is given by $s(t) = A\cos(kt + \phi)$.

Example 3. Find $\dfrac{dy}{dx}$ for the following functions.

(a) $y = e^{x^2} \cos 3x$

(b) $y = \sin \sqrt{x^4 + 1}$

(c) $y = \dfrac{\tan(e^{2x})}{(x^2 + 1)^6}$

Example 4. Find the derivative of the function $y = 2^x$.

The calculation above generalizes to show that

$$\frac{d}{dx}(a^x) = a^x \ln a$$

whenever a is a positive constant. Note that the formula for the derivative of e^x is a special case of this.

Example 5. Find the derivative of the function $y = 5^{\sin x}$

§3.5—Implicit Differentiation

Example 1. Find the equation of the tangent line to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

Solution #1 (Solving for y):

Solution #2 (Differentiating implicitly):

In many of our examples it will not be possible to solve for y, so we'll be forced to use the second method.

Basic procedure for implicit differentiation:

1. Take the derivative of both sides with respect to x. In doing this, we think of y as a function of x, so derivatives of expressions involving y require the Chain Rule.

2. Solve algebraically for $\frac{dy}{dx}$ by collecting all the terms containing $\frac{dy}{dx}$ on one side of the equation and then factoring and dividing.

Implicit differentiation can be used to prove the power rule for rational exponents once it has been proved for integer exponents, as the following example shows.

Example 2. Find $\frac{dy}{dx}$ for the curve defined by $y^3 = x^2$.

Example 3. Find the slope of the tangent line to the curve $3x^4y^2 - 7xy^3 = 4 - 8y$ at the point $(0, 1/2)$.

Example 4. Find $\frac{dy}{dx}$ for the curve $x \cos y + y \cos x = 1$.

§3.6—Inverse Trigonometric Functions and their Derivatives

Even though the trigonometric functions are not one-to-one, we can define inverses for them by restricting their domains to intervals on which the functions are one-to-one. For example, $\sin x$ is one-to-one on the interval $-\pi/2 \leq x \leq \pi/2$ and $\cos x$ is one-to-one on the interval $0 \leq x \leq \pi$. Moreover, these functions cover the full range of y values between -1 and 1 as x runs over these restricted intervals.

- $y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$

- $y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$

- $y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$

Example 1. Evaluate each of the following.

(a) $\sin^{-1}(\frac{1}{2})$ (b) $\cos^{-1}(-\frac{1}{2})$ (c) $\tan^{-1}(1)$

Example 2. Convert the following to algebraic expressions in x.

(a) $\cos(\tan^{-1} \frac{x}{3})$ (b) $\tan(\sin^{-1} x)$

Derivative Formulas. These can all be calculated using right triangles as in Example 2.

$$\frac{d}{dx}(\sin^{-1} x) = \qquad \frac{d}{dx}(\cos^{-1} x) = \qquad \frac{d}{dx}(\tan^{-1} x) =$$

Example 3. Find the derivatives of the following functions.

(a) $y = (\sin^{-1} x)^3$

(b) $y = \tan^{-1}(e^{2x}) - \cos^{-1}(x^4)$

Example 4. Find the equation of the tangent line to the curve $y = 4x \tan^{-1} x$ at $x = 1$.

§3.7—Derivatives of Logarithmic Functions

Suppose that $a > 0$ and $a \neq 1$. The function $y = a^x$ is one-to-one, so it has an inverse, namely $f^{-1}(x) = \log_a x$. The domain of $\log_a x$ is $(0, \infty)$, and the range is $(-\infty, \infty)$

Thus $y = \log_a x \iff a^y = x$. In other words, $\log_a x$ is the power that we must raise a to in order to get x. That is, we have $a^{\log_a x} = x$ for all $x > 0$ and $\log_a a^x = x$ for all x.

Laws of Logarithms:

(1) $\log_a(xy) = \log_a x + \log_a y$ (2) $\log_a(x/y) = \log_a x - \log_a y$ (3) $\log_a x^r = r \log_a x$

The Natural logarithm. We write $\log_e x$ as $\ln x$, so that $y = \ln x \iff e^y = x$.

Derivatives of Logarithmic Functions. We already know how to differentiate a^x. We can use this to find the derivative of $\log_a x$ via implicit differentiation:

Thus we have $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$ and in particular $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Example 1. Compute the derivatives of the following functions.

(a) $y = x^3 \ln x$

(b) $y = \log_2(\cos x)$

(c) $y = \ln |x|$

(d) $y = e^{(\ln x)^2}$

(e) $y = \sqrt{\log_{10}(x^2+1)}$

Remark. By writing $x^n = e^{n \ln x}$ we can use the Chain Rule and the formula for the derivative of $\ln x$ to prove the power rule for any real exponent n.

Example 2 (Logarithmic differentiation). Compute the derivatives of the following functions by first taking the natural log of both sides and then differentiating implicitly.

(a) $y = x^x$

(b) $y = (x^3+1)^5(x^2+3)^4\sqrt{3x+5}$

§3.8—Rates of Change in the Natural and Social Sciences

Recall that the instantaneous rate of change of f with respect to x at a is

$$\left.\frac{dy}{dx}\right|_{x=a} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h},$$

provided the limit exists. This is the limit of the average rates of change of f over smaller and smaller intervals of the form $[a, a+h]$.

Some examples:

$s(t) =$ position $\quad s'(t) =$ velocity

$v(t) =$ velocity $\quad v'(t) = s''(t) =$ acceleration

$Q(t) =$ charge $\quad Q'(t) =$ current

$P(t) =$ population $\quad P'(t) =$ population growth rate

$C(x) =$ cost of producing x units $\quad C'(x) =$ marginal cost

Example 1. The position (in meters) of a particle moving along the s-axis after t seconds is given by $s(t) = \frac{1}{3}t^3 - 2t^2 + 3t$ for $t \geq 0$.

(a) When is the particle moving forward? Backward?

(b) When is the particle's velocity increasing? Decreasing?

(c) What is the total distance traveled by the particle over the first two seconds?

Example 2. Suppose that the cost of producing x washing machines is $C(x) = 2000 + 100x - 0.1x^2$.

(a) Find the marginal cost when 100 washing machines are produced.

(b) Compare the answer to (a) with the cost of producing the 101st machine.

§3.9—Linear Approximations and Differentials

The tangent line to the curve $y = f(x)$ at $x = a$ is given by $y - f(a) = f'(a)(x - a)$, or

$$y = f(a) + f'(a)(x - a).$$

Thus when x is close to a we have

$$f(x) \approx f(a) + f'(a)(x - a).$$

This is called the **linear approximation** or **tangent line approximation** to f at a. The function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Example 1. Compute the linearization of $f(x) = \sqrt{1+x}$ at $a = 0$, and use it to estimate $\sqrt{1.1}$.

Example 2. Use a linear approximation to estimate $\sin(1^\circ)$.

When we are primarily interested in estimating the *change* in a given quantity, it is sometimes more convenient to use **differentials**. If $y = f(x)$ then the differential

$$dy = f'(x)dx$$

gives an estimate for the change in y when we change x by an amount dx. A typical application involves the analysis of error propagation, as in the following example.

Example 3. The edge of a cube is measured at 30 cm, with a possible error of ± 0.1 cm. Estimate the maximum possible error in computing the cube's volume.

§4.1—Related Rates

Suppose that two or more quantities are related by some equation. For instance, if C is the circumference of a circle and r is the radius, then $C = 2\pi r$. As another example, if a and b are the legs of a right triangle with hypotenuse c, then $a^2 + b^2 = c^2$.

If the quantities involved change with time, then we can differentiate both sides of the equation with respect to t to derive a relationship between the rates of change:

$$e.g. \qquad \frac{dC}{dt} = 2\pi\frac{dr}{dt} \qquad \text{or} \qquad 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}.$$

If some of these rates of change are known, then we may be able to use these equations to solve for the unknown rates of change.

Example 1. The radius of a circular oil spill is increasing at a constant rate of 1.5 meters per second. How fast is the area of the spill increasing when the radius is 30 meters?

Example 2. A spherical balloon is being inflated so that its volume is increasing at a constant rate of 5 cubic meters per minute. How fast is its radius increasing when the radius is 3 meters?

Example 3. Boyle's Law states that when a sample of gas is compressed at constant temperature the product of the pressure and the volume remains constant. At a certain instant, the volume of a gas is 600 cubic centimeters, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa per minute. How fast is the volume decreasing at this instant?

Example 4. A ladder 25 feet long is leaning against a vertical wall. The bottom of the ladder is being pulled horizontally away from the wall at a constant rate of 3 feet per second.

(a) How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 15 feet from the wall?

(b) How fast is the angle between the top of the ladder and the wall changing when the bottom is 15 feet from the wall?

Example 5. A tank has the shape of an inverted cone with height 16 meters and base radius 4 meters. Water is being pumped into the tank at a constant rate of 2 cubic meters per minute. How fast is the water level rising when the water is 5 meters deep?

Example 6. You are a camera operator filming an approaching cyclist, who is riding at 10 feet per second along a straight road 30 feet from your camera. How fast does your camera have to pan (in radians per second) to stay pointed at the cyclist when she is 50 feet from your camera?

§4.2—Maximum and Minimum Values

Let f be a function with domain D. We say that f has an **absolute maximum** on D at the point c if $f(x) \leq f(c)$ for all x in D. We say that f has an **absolute minimum** on D at the point c if $f(x) \geq f(c)$ for all x in D.

We say that f has a **local maximum** at c if $f(x) \leq f(c)$ for all x near c. We say that f has a **local minimum** at c if $f(x) \geq f(c)$ for all x near c.

Maxima and minima are sometimes called *extrema*. Absolute extrema are sometimes called *global* extrema, and local extrema are sometimes called *relative* extrema.

Example 1. Identify the coordinates of all absolute and local extrema for the function graphed below on the domain $[0, 10]$.

Notes:

1. Absolute extrema are automatically local extrema.
2. Maximum and minimum *values* refer to the y values on the graph, not the x values at which they occur.
3. Absolute extrema need not exist in general:

Example 2. For each function, give an example of an interval on which there exists

(i) an absolute maximum but no absolute minimum
(ii) an absolute minimum but no absolute maximum
(iii) neither an absolute maximum nor an absolute minimum

(a) $y = x^2$

(b) $y = 1/x^2$

Extreme Value Theorem. If f is continuous on the closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum value in $[a, b]$.

A point in the domain of f where $f' = 0$ or f' does not exist is called a **critical number** of f. Local and absolute extrema can *only* occur at critical numbers or endpoints of the domain.

Example 3. Find the absolute maximum and minimum values of the function $f(x) = x^3 - 12x + 1$ on the interval $[-3, 5]$.

Example 4. Find the absolute maximum and minimum values of the function $f(x) = x^{5/3} - 10x^{2/3}$ on the interval $[-8, 8]$.

Example 5. Find the absolute maximum and minimum values of the function $f(x) = x^2 \ln x$ on the interval $[\frac{1}{2}, \infty)$.

§4.3—Derivatives and the Shapes of Curves

The Mean Value Theorem. Suppose that $y = f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) for which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The picture that makes this "obvious":

Consequences of the MVT. One useful interpretation of the theorem is that there is some point in the interval at which the instantaneous rate of change is equal to the average rate of change. Another important consequence is the following:

Increasing/Decreasing Test:

If $f'(x) > 0$ for all x in some interval, then f is increasing on that interval (*i.e.*, the y values are getting larger).

If $f'(x) < 0$ for all x in some interval, then f is decreasing on that interval (*i.e.*, the y values are getting smaller).

Example 1. Show that the function $f(x) = x^3 + 2x + 4$ is always increasing.

First Derivative Test for Local Extrema: Suppose that c is a critical number of a continuous function f.

1. If f' changes from negative to positive at c, then f has a local minimum at c.
2. If f' changes from positive to negative at c, then f has a local maximum at c.
3. If f' does not change sign at c, then f has no local extremum at c.

Example 2. Find the intervals on which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing, and identify all local extrema of the function.

Concavity

f is concave up $\iff$ f' is increasing $\iff$ $f'' > 0$
f is concave down $\iff$ f' is decreasing $\iff$ $f'' < 0$

Four basic shapes of graphs:

(a) $f' > 0,\ f'' > 0$ (b) $f' > 0,\ f'' < 0$ (c) $f' < 0,\ f'' > 0$ (d) $f' < 0,\ f'' < 0$

A point where the graph of f has a tangent line and where the concavity changes is called an **inflection point** of f. These can only occur where $f'' = 0$ or f'' is undefined.

Example 3. Find the points of inflection of $f(x) = x^3 - 6x^2 + 1$, and determine the intervals on which the curve is concave up and concave down.

Second Derivative Test for Local Extrema: Suppose f'' is continuous and $f'(c) = 0$.

1. If $f''(c) > 0$, then f has a local minimum at $x = c$.
2. If $f''(c) < 0$, then f has a local maximum at $x = c$.
3. If $f''(c) = 0$, then the test gives no information. In this case, we must go back to the first derivative test.

   e.g. $f(x) = x^3$ versus $f(x) = x^4$

Example 4. Use the second derivative test to determine the location of all local maxima and local minima of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. [Compare with Example 2 above.]

Example 5. Sketch the graph of each of the following functions.

(a) $f(x) = x^4 - 4x^3$

(b) $f(x) = x^{2/3}(x - 5)$

§4.5—Indeterminate Forms and L'Hospital's Rule

A general method for evaluating "0/0" or "∞/∞" type limits:

L'Hospital's Rule. Suppose that either

$$\text{(i) } \lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \text{(ii) } \lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty.$$

Then we have $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. Here a can be a real number or $\pm\infty$.

Example 1. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$

(b) $\displaystyle\lim_{x\to 1} \frac{\sin(x-1)}{x^2 + 3x - 4}$

(c) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

Warning: L'Hôpital's Rule does not apply unless (i) or (ii) holds.

For example, $\displaystyle 0 = \lim_{x\to 0} \frac{\sin x}{x+1} \neq \lim_{x\to 0} \frac{\cos x}{1} = 1$

Sometimes it's necessary to apply l'Hôpital's Rule more than once:

Example 2. Evaluate the following limits.

(a) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$

(b) $\lim\limits_{x \to \infty} \dfrac{e^x}{x^4}$

We can sometimes deal with other indeterminate forms like $0 \cdot \infty$, 0^0, $\infty - \infty$, and 1^∞ by converting them to $0/0$ or ∞/∞ and then applying l'Hôpital's Rule.

Example 3. Evaluate the following limits.

(a) $\lim\limits_{x \to 0^+} x \ln x$

(b) $\lim_{x \to 0^+} x^x$

(c) $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

(d) $\lim_{x \to 0^+} (1+x)^{1/x}$

§4.6—Optimization Problems

Example 1. A farmer with 600 feet of fencing wants to construct a rectangular pen and then divide it in half with a fence parallel to one of the sides. What dimensions maximize the area of the pen?

Example 2. You are asked to design a cylindrical can (with top and bottom) of volume 500 cubic centimeters. What dimensions should the can have in order to minimize the amount of metal used?

When arguing that a critical number actually yields the optimal result, we frequently make use of the following principle:

First Derivative Test for Absolute Extrema. Suppose that f is continuous and that c is the *only* critical number of f. If $f(c)$ is a local maximum (resp. minimum), then it is also the absolute maximum (resp. minimum).

Example 3. You are asked to design a new athletic complex in the shape of a rectangle with semi-circular ends. A running track 400 meters long is to go around the perimeter. What dimensions will give the rectangular playing field in the center the largest possible area?

Example 4. A box with no top is to have volume 4 cubic meters, and its base is to be a rectangle twice as long as it is wide. If the material for the bottom costs \$3 per square meter and the material for the sides costs \$1.50 per square meter, find the dimensions that minimize the total cost of constructing the box.

Example 5. You are standing on a sidewalk at the corner of a muddy rectangular field of length 1 mile and width 0.2 miles. You can run along the sidewalk bordering the long side of the field at 8 mph, and you can run through the mud at 5 mph. Find the quickest route to the opposite corner of the field.

Example 6. Find the volume of the largest cylinder that can be inscribed in a sphere of radius R. What percentage of the sphere's volume is occupied by such a cylinder?

§4.8—Antiderivatives

We say that F is an **antiderivative** of f if $F'(x) = f(x)$ for all x.

x^2 and $x^2 + 1$ are antiderivatives of $2x$

$\sin x$ and $\sin x - 17$ are antiderivatives of $\cos x$

If F is any antiderivative of f, then it follows from the Mean Value Theorem that the most general antiderivative of f is $F(x) + C$, where C is an arbitrary constant.

Example 1. Find the most general antiderivative for each of the following functions.

(a) $f(x) = x^2 + 2\cos x$

(b) $f(x) = \sin x + x^{-6}$

(c) $f(x) = 5\sqrt{x} + 3\sec^2 x$

(d) $f(x) = \cos 2x - e^{3x}$

(e) $f(x) = \dfrac{5}{x} + \dfrac{10}{x^2 + 1} - 3^x$

Example 2. Suppose that $f'(x) = 3x^2$ and $f(1) = 5$. Find a formula for $f(x)$.

Example 3. A particle's acceleration is given by $a(t) = 5 + 4t - 2t^2$, and its initial velocity and position are $v(0) = 3$ and $s(0) = 10$. Find formulas for $v(t)$ and $s(t)$.

Example 4. Solve the initial value problem

$$y'''(x) = \cos 2x,\ \ y''(0) = 3,\ \ y'(0) = 1,\ \ y(0) = 2.$$

§5.1—Areas and Distances

Example 1. Estimate the area of the region bounded by the curve $y = x^2$ and the x-axis between $x = 0$ and $x = 2$ by approximating the region with 4 rectangles of equal width whose heights are determined using

(a) left endpoints

(b) right endpoints

(c) midpoints

Using a larger number of rectangles gives a better estimate of the area, and we define the exact area to be the limit of these approximations as the number of rectangles tends to infinity.

Example 2. Let A be the exact area bounded by the curve $y = x^2$ and the x-axis between $x = 0$ and $x = 2$. Express A as a limit of finite sums.

Example 3. A car's velocity during a 1-hour period is measured at 12-minute intervals:

time (hours)	0	0.2	0.4	0.6	0.8	1.0
velocity (miles per hour)	66	75	78	82	79	74

Estimate the total distance traveled by the car during the hour using

(a) left endpoints

(b) right endpoints

Example 4. If the velocity of the car in the previous example is given by $v(t)$, express the exact distance traveled during the 60 minutes as a limit of finite sums.

§5.2—The Definite Integral

To approximate the area between $y = f(x)$ and the x-axis on the interval $[a, b]$, we divide into n subintervals of width

$$\Delta x = \frac{b-a}{n}.$$

The ith subinterval is the interval $[x_{i-1}, x_i]$, where $x_i = a + i\Delta x$. For each i, we choose a sample point x_i^* in the interval $[x_{i-1}, x_i]$ and use a rectangle of height $f(x_i^*)$ and width Δx to approximate the area under that portion of the curve.

The **Riemann sum**

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

approximates the total area under the curve on the interval $[a, b]$. We get the exact area by letting $n \to \infty$, which gives the **definite integral** of f from a to b:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided that this limit exists and has the same value for every choice of the sample points. Here the function f is called the **integrand** and the numbers a and b are the **limits of integration**.

Example 1. Consider a partition of the interval $[1, 4]$, with sample points x_i^*. Express the following limits of Riemann sums as definite integrals.

(a) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{x_i^*}\right) \Delta x$

(b) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^*)^2 e^{x_i^*} \Delta x$

When f is continuous, the definite integral always exists and can be calculated using any choice of sample points. In this case, we often choose the right-hand endpoints for convenience, which gives the simpler formula

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$.

Example 2. Use the summation formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

to calculate the following definite integrals directly from the definition.

(a) $\displaystyle\int_0^3 x\,dx$

(b) $\displaystyle\int_0^2 x^2\,dx$

In general, calculating definite integrals directly from the definition as a limit of Riemann sums can be difficult, or at least tedious. Fortunately, we'll learn a better method in Sections 5.3 and 5.4.

If $f(x)$ takes on both positive and negative values on $[a, b]$, then the definite integral gives the "signed area" under the curve. That is, areas above the x-axis are counted positively, and areas below the x-axis are counted negatively.

Example 3. Evaluate the following by interpreting the integral in terms of signed area under the curve.

(a) $\displaystyle\int_0^{2\pi} \sin x \, dx$

(b) $\displaystyle\int_{-2}^{3} x \, dx$

Properties of the definite integral

- Conventions: $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ and $\displaystyle\int_a^a f(x)\,dx = 0$
- Linearity: $\displaystyle\int_a^b (kf(x) \pm mg(x))\,dx = k\int_a^b f(x)\,dx \pm m\int_a^b g(x)\,dx$ (k, m constant)
- Additivity: $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- Domination: If $f(x) \leq g(x)$ for all x in $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Example 4. Suppose that $\displaystyle\int_0^1 f(x)\,dx = 2$ and that $f(x) \leq 4$ for all x in $[1, 3]$. What is the largest possible value that the integral $\displaystyle\int_0^3 f(x)\,dx$ could have?

§5.3—Evaluating Definite Integrals

It turns out that the key to evaluating definite integrals efficiently is finding an antiderivative for the integrand. We actually observed a special case of this in Example 4 of Section 5.1 when we saw that the area under a velocity graph gives the net change in position. More generally, if f has an antiderivative F, then we can view f as a rate of change of F and apply the same reasoning to establish the following:

Evaluation Theorem. If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is really Part 2 of the Fundamental Theorem of Calculus, which we'll study in the next section. It may be interpreted as saying that the definite integral of a rate of change gives the total change.

Example 1. Use the Evaluation Theorem to calculate each of the following.

(a) $\displaystyle\int_0^2 x^2\,dx$

(b) $\displaystyle\int_0^{\pi/2} \cos x\,dx$

Recall that if F is any antiderivative of f, then the most general antiderivative of f is $F(x) + C$, where C is an arbitrary constant. The set of all antiderivatives of f is denoted $\displaystyle\int f(x)\,dx$ and is called the **indefinite integral** of f with respect to x. For example,

$$\int 2x\,dx = x^2 + C \qquad \text{and} \qquad \int \cos x\,dx = \sin x + C.$$

This notation is suggested by the relationship between definite integrals and antiderivatives given in the Evaluation Theorem. However, it's important to remember that a definite integral represents a number while an indefinite integral represents a family of functions.

Example 2. Evaluate each of the following.

(a) $\int (3 + x\sqrt{x})\, dx$

(b) $\int_0^1 (3 + x\sqrt{x})\, dx$

Partial Table of Indefinite Integrals (see back reference pages for a more complete list)

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int \sin x\, dx = -\cos x + C \qquad \int \cos x\, dx = \sin x + C$$

$$\int e^x\, dx = e^x + C \qquad \int a^x\, dx = \frac{a^x}{\ln a} + C \quad (a \neq 1,\ a > 0)$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$$

Example 3. Evaluate each of the following.

(a) $\int_0^1 \frac{dx}{1+x^2}$

(b) $\int_1^3 \left(\frac{2}{x} - e^x\right)\, dx$

§5.4—The Fundamental Theorem of Calculus

Part 1. Suppose that f is continuous on $[a, b]$, and let $F(x) = \int_a^x f(t)\,dt$. Then

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Why is this true?

Interpretation: Differentiation and integration are "inverse" operations, *i.e.*, $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

Example 1. Calculate each of the following derivatives.

(a) $\displaystyle \frac{d}{dx}\int_1^x \frac{\sin t}{t}\,dt$

(b) $\displaystyle \frac{d}{dx}\int_4^{x^3} \sqrt{1+t^2}\,dt$

(c) $\displaystyle \frac{d}{dx}\int_{-1}^{e^{2x}} \cos^4 t\,dt$

(d) $\displaystyle \frac{d}{dx}\int_{\tan^2 x}^{5} \ln t\,dt$

Part 2. If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Interpretation: The definite integral of a rate of change gives the total change.

Example 2. Evaluate each of the following definite integrals.

(a) $\displaystyle\int_0^3 e^{5x}\,dx$ (b) $\displaystyle\int_0^1 \sqrt{x}(x^2 + 3)\,dx$

Example 3. What is wrong with the following calculation?

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}.$$

Example 4. For what values of x is the function $F(x) = \displaystyle\int_0^x \frac{1}{1 + t + t^2}\,dt$ concave up?

§5.5—The Substitution Rule

In earlier sections, we obtained formulas like

$$\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C \qquad \text{and} \qquad \int e^{5x} \, dx = \frac{1}{5} e^{5x} + C$$

by mentally attempting to reverse the effect of the chain rule. A more systematic approach is to substitute a new variable for the inner function. For instance, if we let $u = 2x$ in the first integral above, then $du = 2dx$, and thus $dx = \frac{1}{2}du$, so we get

$$\int \cos 2x \, dx = \int (\cos u) \frac{1}{2} du = \frac{1}{2} \int \cos u \, du = \frac{1}{2} \sin u + C = \frac{1}{2} \sin 2x + C.$$

In general, we can evaluate $\displaystyle\int f(g(x))g'(x) \, dx$ by substituting $u = g(x)$ and $du = g'(x) \, dx$.

Example 1. Evaluate the following indefinite integrals.

(a) $\displaystyle\int 2xe^{x^2} \, dx$

(b) $\displaystyle\int \sqrt{3x+4} \, dx$

(c) $\displaystyle\int x^4 \cos(x^5) \, dx$

(d) $\displaystyle\int \frac{(1 + \ln x)^{10}}{x} \, dx$

Example 2. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(b) $\displaystyle\int \frac{\sec^2 x}{1+\tan x}\,dx$

(c) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

Substitution in Definite Integrals: $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

Example 3. Use substitution to evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

(b) $\displaystyle\int_0^{\pi/2} (1+\sin^3 x)\cos x\,dx$

(c) $\displaystyle\int_2^4 \frac{dx}{x \ln x}$

(d) $\displaystyle\int_{\pi/4}^{\pi/2} e^{\cot\theta} \csc^2\theta \, d\theta$

Symmetry: If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$. If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Example 4. Use symmetry to evaluate the following integrals.

(a) $\displaystyle\int_{-1}^{1} (x^4 + 3x^2 + 1)\,dx$

(b) $\displaystyle\int_{-2}^{2} (x^5 - 5x^3 + 12x - \sin x)\,dx$

Areas between curves: If $f(x) \geq g(x)$ for all x in $[a, b]$ then the area between f and g from $x = a$ to $x = b$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

Example 5. Find the area bounded between the curve $y = x^2$ and the line $y = x$.