# HUA-TYPE ITERATION FOR MULTIDIMENSIONAL WEYL SUMS

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Abstract. We develop Weyl differencing and Hua-type lemmata for a class of multi-dimensional exponential sums. We then apply our estimates to bound the number of variables required to establish an asymptotic formula for the number of solutions of a system of diophantine equations arising from the study of linear spaces on hypersurfaces. For small values of the degree and dimension, our results are superior to those stemming from the author's earlier work on Vinogradov's mean value theorem.

## 1. Introduction

The considerable machinery associated with the Hardy-Littlewood method is quite successful in estimating the number of integral points in a box that satisfy a diagonal equation or a system of diagonal equations in many variables. Although the method is now nearly a century old, the recent innovations of Wooley [39], [40] on Vinogradov's mean value theorem and Waring's problem demonstrate that substantial progress is still being made on even the most classical questions. As a consequence of these breakthroughs, one now obtains the expected asymptotic formula for the number of solutions of the additive equation

$$c_1x_1^k + \cdots + c_sx_s^k = 0, \tag{1.1}$$

where $c_1, \ldots, c_s$ are nonzero integers, provided that $s \geq 2k^2 - 2\lfloor \log_2 k \rfloor$. This may be compared with bounds of the shape $Ck^2 \log k$ stemming from work of Vinogradov [32], Hua [19], Wooley [34], and Ford [13]. For smaller $k$, methods based on refinements of Weyl differencing and Hua-type lemmata (see for example Vaughan [27], [28], Heath-Brown [16], and Boklan [5]) lead to superior conclusions. Finally, the iterative method of Vaughan and Wooley [29], [31], [33] delivers asymptotic lower bounds of the correct order of magnitude for the number of solutions of (1.1), subject to local solubility hypotheses.

For non-diagonal forms, the circle method has been less successful in achieving reasonable bounds on the number of variables required, even to demonstrate the existence of a single non-trivial solution. While some exceptional cases involving cubic equations ([10], [11], [15], [17], [18]) and sums of non-degenerate binary forms ([8], [9], [38]) have been handled rather successfully, the general exponential sum estimates available from work of Birch [4] and Schmidt [26], for example, typically yield bounds growing at least exponentially in the degrees of the underlying forms. An alternative path towards the existence of solutions to non-diagonal systems, originating in the work of Brauer [6] and Birch [3], is to employ a diagonalization argument that involves inductively producing linear spaces of solutions. More recent work has led to bounds of an explicit nature in various situations (see for instance [12], [21], [35], [37]) for the number of variables required to obtain such

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spaces, but the most general results still exhibit at least exponential growth in the degrees of the underlying forms.

In the diagonal situation, the perspective on linear spaces is somewhat different. Here one finds that the linear spaces of affine dimension $d$ up to a certain height $P$ are in correspondence with certain equivalence classes of solutions to the diophantine system

$$c_1x_{11}^{j_1}\cdots x_{1d}^{j_d}+\cdots+c_sx_{s1}^{j_1}\cdots x_{sd}^{j_d}=0 \qquad (j_1+\cdots+j_d=k), \tag{1.2}$$

lying in the box $[1,P]^{sd}$. This leads to a problem of independent interest, for we now have a natural example of a non-diagonal system with enough specialized structure to attempt to adapt the techniques that work so well in the additive case. Systems of the shape (1.2) were first investigated in detail by Arkhipov, Karatsuba, and Chubarikov [1], who adapted Vinogradov's methods to show that a bound of the shape $s \geq Ck^{d+1}\log k$ suffices to establish asymptotic formulas under a local solubility hypothesis. The author [22], [23], [25] has extended the newer iterative methods involving smooth numbers (see [36]) to obtain asymptotic lower bounds of the correct order of magnitude with fewer variables and has also developed a multidimensional generalization of Vinogradov's mean value theorem [24] that sharpens the results of [1]. The purpose of the present paper is to demonstrate how classical methods of Weyl and Hua can be used to attack the system (1.2) directly for smaller values of $k$ and $d$.

The system (1.2) arising from linear spaces on (1.1) is actually unnecessarily specialized for the analysis since the coefficients $c_1,\dots,c_s$ are the same in each equation. At least from the point of view of establishing a Hasse principle, there is no difficulty in allowing more general coefficient arrays. We adopt the shorthand notation $\mathbf{x}^{\mathbf{j}}=x_1^{j_1}\cdots x_d^{j_d}$ and $|\mathbf{j}|=j_1+\cdots+j_d$, and let $\{c_{i\mathbf{j}}\}$ be nonzero integers. Write $N_{s,k,d}(P)=N_{s,k,d}(P;\mathbf{c})$ for the number of solutions of the system

$$c_{1\mathbf{j}}\mathbf{x}_1^{\mathbf{j}}+\cdots+c_{s\mathbf{j}}\mathbf{x}_s^{\mathbf{j}}=0 \qquad (|\mathbf{j}|=k), \tag{1.3}$$

with $\mathbf{x}_1,\dots,\mathbf{x}_s\in[-P,P]^d\cap\mathbb{Z}^d$, and write

$$\ell=\binom{k+d-1}{d-1}$$

for the number of equations in the system (1.3). Our methods deliver the following conclusion for the systems associated to lines on cubic hypersurfaces, which improves on earlier work of the author [22].

**Theorem 1.1.** *When $s\geq 29$, one has $N_{s,3,2}(P)=\sigma P^{2s-12}+O(P^{2s-12-\delta})$ for some positive constants $\sigma$ and $\delta$ depending on $s$ and $\mathbf{c}$.*

We note for comparison that the argument of [22] establishes the weaker conclusion $N_{s,3,2}(P)\gg P^{2s-12}$ under the stronger hypothesis that $s\geq 55$. We mention that the constant $\sigma$ appearing in (1.6) represents the usual product of local densities. Specifically, one has $\sigma=\mathfrak{J}_{s,3,2}(\mathbf{c})\mathfrak{S}_{s,3,2}(\mathbf{c})$, where $\mathfrak{J}=\mathfrak{J}_{s,k,d}(\mathbf{c})$ and $\mathfrak{S}=\mathfrak{S}_{s,k,d}(\mathbf{c})$ are the singular integral and singular series defined by (4.1) and (4.2) below.

In order to estimate $N_{s,k,d}(P)$ via the Hardy-Littlewood method, one needs upper bounds for the number of solutions of an auxiliary symmetric system. We write $I_{s,k,d}(P)$ for the number of solutions of the system

$$\sum_{i=1}^{s}(\mathbf{x}_i^{\mathbf{j}}-\mathbf{x}_{s+i}^{\mathbf{j}})=0 \qquad (|\mathbf{j}|=k) \tag{1.4}$$

with $\mathbf{x}_1, \ldots, \mathbf{x}_{2s} \in [1, P]^d \cap \mathbb{Z}^d$, and observe that

$$I_{s,k,d}(P) = \int_{[0,1)^\ell} |f(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha},$$

where

$$f(\boldsymbol{\alpha}) = \sum_{\mathbf{x} \in [1,P]^d} e\bigg( \sum_{|\mathbf{j}|=k} \alpha_{\mathbf{j}} \mathbf{x}^{\mathbf{j}} \bigg). \tag{1.5}$$

The bulk of our effort is devoted to obtaining Weyl-type estimates for the exponential sum (1.5) and Hua-type estimates for the mean values (1.4). For pairs $(k, d) \neq (3, 2)$, when information about local solubility is not available, our methods nevertheless lead to a strong Hasse-type principle for the system (1.3). We let $H(k, d)$ denote the smallest integer $s$ such that, whenever the system (1.3) possesses a non-singular real solution and and non-singular $p$-adic solutions for each prime $p$, one has the asymptotic formula

$$N_{s,k,d}(P) = \mathfrak{J}\mathfrak{S} P^{sd-k\ell} + O(P^{sd-k\ell-\delta}), \tag{1.6}$$

where $\mathfrak{J}\mathfrak{S} > 0$ and where $\delta$ is a positive number depending at most on $s$, $k$, $d$, and $\mathbf{c}$. When $k = 3$ or $d = 2$, rather straightforward strategies lead to the following estimates.

**Theorem 1.2.** *For $k \geq 4$, one has*

$$H(k, 2) \leq k(k-1)2^{k-2} + 2k(k+1) + 1,$$

*while for $d \geq 2$, one has*

$$H(3, d) \leq \min\{2d^3 + 6d^2 - 20d + 29,\ \tfrac{5}{3}d^3 + 5d^2 + \tfrac{10}{3}d + 1\}.$$

In particular, we note that the first estimate of Theorem 1.2 produces the bounds $H(4, 2) \leq 89$, $H(5, 2) \leq 221$, $H(6, 2) \leq 565$, and $H(7, 2) \leq 1457$, although the latter two bounds may be susceptible to improvement by combining the methods of [24] and [40]. The first alternative in the second estimate of Theorem 1.2 gives $H(3, 3) \leq 77$, $H(3, 4) \leq 173$, $H(3, 5) \leq 329$, and $H(3, 6) \leq 557$, while the second alternative becomes superior for $d \geq 7$ and delivers the bounds $H(3, 7) \leq 841$, $H(3, 8) \leq 1201$, $H(3, 9) \leq 1651$, $H(3, 10) \leq 2201$, and $H(3, 11) \leq 2861$. When $d \geq 12$, the conclusions stemming from our methods are again likely to be inferior to those obtainable by applying Wooley's efficient congruencing [40] to multidimensional Vinogradov-type systems of the shape studied in [24]. Here one seeks bounds for $J_{s,k,d}(P)$, the number of solutions of the augmented system

$$\sum_{i=1}^{s} (\mathbf{x}_i^{\mathbf{j}} - \mathbf{x}_{s+i}^{\mathbf{j}}) = 0 \qquad (1 \leq |\mathbf{j}| \leq k) \tag{1.7}$$

with $\mathbf{x}_1, \ldots, \mathbf{x}_{2s} \in [1, P]^d \cap \mathbb{Z}^d$. Such estimates can be used to bound $I_{s,k,d}(P)$ by summing over the equations with $1 \leq |\mathbf{j}| \leq k - 1$ or possibly by adapting the strategy of Wooley [39] to engineer a more efficient relationship; the latter is a project that we intend to pursue in a separate paper. One can also adapt the method of Ford [13] to relate the mean values associated with (1.4) and (1.7), but it turns out that this approach yields conclusions weaker than ours in all cases under consideration.

Finally, as an illustration of the scope of our methods, we record bounds obtainable for other small pairs $(k, d)$ with $k \geq 4$ and $d \geq 3$, in which cases the Hua-type machinery is more robust.

**Theorem 1.3.** *One has the bounds*

$$H(4,3) \leq 293, \quad H(4,4) \leq 809, \quad \textit{and} \quad H(5,3) \leq 1093.$$

We remark that the bounds $H(4,5) \leq 1797$, $H(4,6) \leq 3277$, $H(5,4) \leq 3283$, and $H(6,3) \leq 3741$ are also attainable by our methods, but we expect that significantly better results may be obtained by adapting the new Vinogradov-type techniques of [40]. This is another project that we defer to a future paper.

Throughout, we suppose that $k \geq 2$ and $d \geq 2$ and that $P$ is sufficiently large in terms of $s$, $k$, $d$, and $\mathbf{c}$. We further adopt the convention that statements involving $\varepsilon$ are meant to hold for all positive values of $\varepsilon$. We begin in Section 2 by developing our Weyl differencing procedure and addressing a difficulty concerning its applicability on a standard set of minor arcs. Then in Section 3 we apply the differencing machinery to iteratively construct estimates for the mean values $I_{s,k,d}(P)$ in a style reminiscent of Hua's Lemma. A new feature here is that the iteration takes place both with respect to $s$ and $d$. Finally, in Section 4, we outline an application of the circle method to deduce our main conclusions.

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## 2. Weyl differencing and quasi-major arcs

In this section, we describe a simple differencing algorithm for the exponential sum (1.5) that allows us to isolate a small subset of the coefficients $\alpha_{\mathbf{j}}$ for further analysis. The main idea is to use Cauchy's inequality to difference with respect to only one of the $d$ variables at each stage. Here we find it convenient to introduce a *differencing vector* $\mathbf{i} = (i_1, \dots, i_d)$ with $|\mathbf{i}| \leq k-1$ to indicate that we difference $i_l$ times with respect to the variable $x_l$, for $1 \leq l \leq d$. We further write $\mathscr{J}(\mathbf{i})$ for the set of $\mathbf{j}$ with $|\mathbf{j}| = k$ with $j_l \geq i_l$ for each $l$; thus $\mathscr{J}(\mathbf{i})$ represents the set of indices $\mathbf{j}$ for which $\alpha_{\mathbf{j}}$ appears explicitly in the resulting difference polynomial. Finally, it is convenient to write $\mathbf{e}_l = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^d$ for the vector with 1 in the $l$th position and zeros elsewhere. Our process is encapsulated in the following lemma.

**Lemma 2.1.** *Whenever $1 \leq j \leq k-1$ and $i_1, \dots, i_d$ are non-negative integers satisfying $i_1 + \cdots + i_d = j$, one has*

$$|f(\boldsymbol{\alpha})|^{2^j} \ll P^{d(2^j-1)-j} \sum_{\mathbf{h} \in (-P,P)^j} \sum_{\mathbf{x} \in \mathscr{B}_{\mathbf{h}}(P)} e\Bigg( h_1 \cdots h_j \sum_{\mathbf{j} \in \mathscr{J}(\mathbf{i})} \alpha_{\mathbf{j}} p_{\mathbf{ij}}(\mathbf{x}; \mathbf{h}) \Bigg),$$

*where $\mathscr{B}_{\mathbf{h}}(P)$ is some box contained in $[1,P]^d$, and where*

$$p_{\mathbf{ij}}(\mathbf{x}; \mathbf{h}) = \frac{j_1! \cdots j_d!}{(j_1 - i_1)! \cdots (j_d - i_d)!} \mathbf{x}^{\mathbf{j}-\mathbf{i}} + q_{\mathbf{ij}}(\mathbf{x}; \mathbf{h})$$

*for some polynomial $q_{\mathbf{ij}}$ with integer coefficients of total degree $k-j-1$ in $\mathbf{x}$ and degree at most $j_l - i_l$ in $x_l$ for $1 \leq l \leq d$.*

*Proof.* We proceed by induction on $j$, noting that for $j=1$ and $\mathbf{i}=\mathbf{e}_l$ an application of Cauchy's inequality gives

$$|f(\boldsymbol{\alpha})|^2 \le P^{d-1} \sum_{|h|<P} \sum_{\mathbf{x}\in\mathscr{B}_h(P)} e\biggl(\sum_{|\mathbf{j}|=k} \alpha_{\mathbf{j}}\bigl((\mathbf{x}+h\mathbf{e}_l)^{\mathbf{j}}-\mathbf{x}^{\mathbf{j}}\bigr)\biggr)$$

for some box $\mathscr{B}_h(P)\subseteq[1,P]^d$. It is clear that $(\mathbf{x}+h\mathbf{e}_l)^{\mathbf{j}}-\mathbf{x}^{\mathbf{j}}=0$ whenever $j_l=0$, and hence we may replace the sum over $|\mathbf{j}|=k$ by one over $\mathbf{j}\in\mathcal{J}(\mathbf{e}_l)$. Moreover, whenever $\mathbf{j}_l\ge 1$ we find that

$$(\mathbf{x}+h\mathbf{e}_l)^{\mathbf{j}}-\mathbf{x}^{\mathbf{j}} = h\biggl(j_l\mathbf{x}^{\mathbf{j}-\mathbf{e}_l} + \sum_{m=2}^{j_l}\binom{j_l}{m}h^{m-1}\mathbf{x}^{\mathbf{j}-m\mathbf{e}_l}\biggr), \tag{2.1}$$

which shows that the polynomial $p_{\mathbf{e}_l\mathbf{j}}(\mathbf{x};h)$ has the required properties. Next suppose the result holds for some $j\ge 1$ and all $\mathbf{i}$ satisfying $|\mathbf{i}|=j$. Then if $|\mathbf{i}|=j+1$, we select some $l$ for which $i_l>0$ and write $\mathbf{i}'=\mathbf{i}-\mathbf{e}_l$. Further set $\mathbf{x}'=(x_1,\dots,x_{l-1},x_{l+1},\dots,x_d)$, and write $\mathcal{I}_{\mathbf{h},l}(P)$ for the projection of $\mathscr{B}_{\mathbf{h}}(P)$ onto the $x_l$ axis. Then the induction hypothesis in conjunction with Cauchy's inequality yields

$$\begin{aligned}|f(\boldsymbol{\alpha})|^{2^{j+1}} &\ll P^{2d(2^j-1)-2j}\cdot P^{j+d-1}\sum_{\mathbf{h},\mathbf{x}'}\biggl|\sum_{x_l\in\mathcal{I}_{\mathbf{h},l}(P)} e\biggl(h_1\cdots h_j\sum_{\mathbf{j}\in\mathcal{J}(\mathbf{i}')}\alpha_{\mathbf{j}}p_{\mathbf{i}'\mathbf{j}}(\mathbf{x};\mathbf{h})\biggr)\biggr|^2\\ &= P^{d(2^{j+1}-1)-(j+1)}\sum_{\mathbf{h},h_{j+1},\mathbf{x}} e\biggl(h_1\cdots h_j\sum_{\mathbf{j}\in\mathcal{J}(\mathbf{i}')}\alpha_{\mathbf{j}}\bigl(p_{\mathbf{i}'\mathbf{j}}(\mathbf{x}+h_{j+1}\mathbf{e}_l;\mathbf{h})-p_{\mathbf{i}'\mathbf{j}}(\mathbf{x};\mathbf{h})\bigr)\biggr)\end{aligned}$$

where $p_{\mathbf{i}'\mathbf{j}}$ is as in the statement of the lemma. Here the summations are taken over $(\mathbf{h},h_{j+1})\in(-P,P)^{j+1}$ and $\mathbf{x}$ lying in some box $\mathscr{B}_{\mathbf{h},h_{j+1}}(P)\subseteq[1,P]^d$. Now an application of (2.1) to each term of $p_{\mathbf{i}'\mathbf{j}}$ shows that

$$p_{\mathbf{i}'\mathbf{j}}(\mathbf{x}+h_{j+1}\mathbf{e}_l;\mathbf{h})-p_{\mathbf{i}'\mathbf{j}}(\mathbf{x};\mathbf{h}) = h_{j+1}p_{\mathbf{i}\mathbf{j}}(\mathbf{x};\mathbf{h},h_{j+1}), \tag{2.2}$$

where $p_{\mathbf{i}\mathbf{j}}$ satisfies the hypotheses of the lemma with $j$ replaced by $j+1$ whenever $\mathbf{j}\in\mathcal{J}(\mathbf{i})$. Moreover, the left-hand side of (2.2) is identically zero whenever $\mathbf{j}\in\mathcal{J}(\mathbf{i}')\setminus\mathcal{J}(\mathbf{i})$. This completes the induction, and the lemma is proved. ☐

As in the familiar one-dimensional situation, the case $j=k-1$ of the above result is particularly useful for exploiting information concerning rational approximations to the $\alpha_{\mathbf{j}}$, and we obtain a result bearing strong resemblance to Weyl's inequality.

**Lemma 2.2.** *If $(q_{\mathbf{j}},a_{\mathbf{j}})=1$ and $|\alpha_{\mathbf{j}}-a_{\mathbf{j}}/q_{\mathbf{j}}|\le q_{\mathbf{j}}^{-2}$ for some $\mathbf{j}$ with $|\mathbf{j}|=k$ then one has*

$$f(\boldsymbol{\alpha})\ll P^{d+\varepsilon}(q_{\mathbf{j}}^{-1}+P^{-1}+q_{\mathbf{j}}P^{-k})^{2^{1-k}}.$$

*Proof.* To isolate the coefficient $\alpha_{\mathbf{j}}$, we choose any $t$ for which $j_t>0$ and apply Lemma 2.1 with differencing vector $\mathbf{i}=\mathbf{j}-\mathbf{e}_t$. Thus we obtain

$$|f(\boldsymbol{\alpha})|^{2^{k-1}} \ll P^{d(2^{k-1}-1)-k+1}\sum_{\mathbf{h}\in(-P,P)^{k-1}}\biggl|\sum_{\mathbf{x}\in\mathscr{B}_{\mathbf{h}}(P)} e(h_1\cdots h_{k-1}j_1!\cdots j_d!(\alpha_{\mathbf{j}_1}x_1+\cdots+\alpha_{\mathbf{j}_d}x_d))\biggr|,$$

where $\mathbf{j}_l=\mathbf{j}-\mathbf{e}_t+\mathbf{e}_l$, so that in particular we have $\mathbf{j}=\mathbf{j}_t$. Upon summing the geometric progression on $x_t$ and splitting off the $\mathbf{h}$ for which $h_1\cdots h_{k-1}=0$, it now follows from a standard estimate for the divisor function that

$$|f(\boldsymbol{\alpha})|^{2^{k-1}} \ll P^{d(2^{k-1}-1)-k+1}\Bigg(P^{d+k-2} + P^{d-1+\varepsilon} \sum_{1\le h\le \mathbf{j}!P^{k-1}} \min\{P, ||h\alpha_{\mathbf{j}}||^{-1}\}\Bigg),$$

where we have written $\mathbf{j}! = j_1!\cdots j_d!$. The lemma now follows by applying Lemma 2.2 of Vaughan [30] as in the usual proof of Weyl's inequality. ☐

A direct application of Lemma 2.2 on a standard set of minor arcs, in which the complementary major arcs demand that each $\alpha_{\mathbf{j}}$ is well-approximated using a common denominator $q \le P$, has the potential to produce very weak Weyl estimates (see for example Lemma 4.2 of [22]) unless one can obtain excellent control over the least common multiple of the various $q_{\mathbf{j}}$ when the Weyl sum is large. The arguments of Baker [2], Theorems 4.3 and 5.1, handle this type of problem in the one-dimensional case, but we have been unable to adapt those methods to the present context. Fortunately, we are able to circumvent this issue by first dealing with a considerably larger set of quasi-major arcs on which the expected Weyl estimate fails. Before proceeding with this approach, it is useful to have available a preliminary mean value estimate that applies to any subset of the equations (1.4).

**Lemma 2.3.** *Suppose that $\mathcal{T}$ is any subset of the set of indices $\mathbf{j}$ with $|\mathbf{j}| = k$, and write $T = |\mathcal{T}|$. Further let $I_{s,k,d}(P;\mathcal{T})$ denote the number of solutions of the system*

$$\sum_{i=1}^{s}(\mathbf{x}_i^{\mathbf{j}} - \mathbf{x}_{s+i}^{\mathbf{j}}) = 0 \qquad (\mathbf{j}\in\mathcal{T}) \tag{2.3}$$

*with $\mathbf{x}_1,\dots,\mathbf{x}_{2s}\in[1,P]^d$. Then whenever $s \ge T$, one has*

$$I_{s,k,d}(P;\mathcal{T}) \ll P^{2sd-T}.$$

*Proof.* We classify solutions according to the size of the largest non-zero sub-determinant present in the Jacobian. Write $\mathrm{Jac}(\mathbf{x})$ for the $T\times 2sd$ Jacobian matrix of the system (2.3), which we arrange in $2s$ blocks of $d$ variables each, so that the $i$th block depends only on the variables $\mathbf{x}_i$. We let $V_m$ denote the number of solutions counted by $I_{s,k,d}(P;\mathcal{T})$ for which $\mathrm{Jac}(\mathbf{x})$ contains an $m\times m$ sub-matrix with nonzero determinant but no $(m+1)\times(m+1)$ sub-matrix with nonzero determinant. Then one clearly has

$$I_{s,k,d}(P;\mathcal{T}) \ll \sum_{m=1}^{T} V_m.$$

First suppose that $1 \le m < T$, and consider a solution counted by $V_m$. There are $\binom{2sd}{m}\binom{T}{m} \ll 1$ possible $m\times m$ sub-matrices of $\mathrm{Jac}(\mathbf{x})$, and we consider one such matrix $A_m$ with nonzero determinant. There are $O(P^{md})$ choices for the at most $md$ different variables appearing in $A_m$, and given any such choice, there are at least $2s-m$ remaining blocks of variables disjoint from those appearing in $A_m$. Since $m < T$, we may select a new row $R_{m+1}$ of $\mathrm{Jac}(\mathbf{x})$ not appearing in $A_m$. Then within the block represented by a variable $\mathbf{x}_i$ disjoint from $A_m$, we may select a column $C_{m+1}$ for which some variable $x_{il}$ appears explicitly in row $R_{m+1}$. By adjoining $R_{m+1}$ and $C_{m+1}$ to $A_m$ in the obvious way, we produce an $(m+1)\times(m+1)$ sub-matrix of $\mathrm{Jac}(\mathbf{x})$, whose determinant is necessarily zero by definition of $V_m$. But no two entries of a given column contain the same monomial, and hence expanding by minors along $C_{m+1}$ shows that this determinant is a non-trivial polynomial in some non-empty subset of the variables $x_{i1},\dots,x_{id}$. Hence there are $O(P^{d-1})$ choices for $\mathbf{x}_i$, and repeating this argument for the remaining blocks of variables disjoint from $A_m$ yields the estimate

$$V_m \ll P^{md}(P^{d-1})^{2s-m} \ll P^{2sd-2s+m} \ll P^{2sd-2T+m} \ll P^{2sd-T-1}$$

for $1 \le m < T$. Finally, consider a solution counted by $V_T$. There are again $O(1)$ choices for the $T$ columns of $\mathrm{Jac}(\mathbf{x})$ in which a non-zero sub-determinant occurs, and we note that at least $T$ different variables appear explicitly in any such choice of columns. Thus, after fixing all but $T$ variables, we obtain a system of $T$ polynomial equations in $T$ unknowns, and we may apply a generalization of Bézout's Theorem (see Theorem 7.7 in Chapter 1 of Hartshorne [14]) in combination with the Implicit Function Theorem to deduce that the number of non-singular solutions is bounded by the product of the degrees. Thus we conclude that $V_T \ll P^{2sd-T}$, and this completes the proof. □

We are now able to handle the quasi-major arcs described following the proof of Lemma 2.2. Let $\mathfrak{M}$ denote the set of $\boldsymbol{\alpha} \in [0,1)^\ell$ with the property that $|f(\boldsymbol{\alpha})| \ge P^{d-2^{1-k}+\tau}$ for some $\tau > 0$, and write $\mathfrak{m} = [0,1)^\ell \setminus \mathfrak{M}$. The following lemma allows us to obtain mean value estimates towards the end of our iterative process by performing a Hardy-Littlewood dissection.

**Lemma 2.4.** *Whenever $s \ge (2^{k-2}+1)\ell$, one has*

$$\int_{\mathfrak{M}} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \ll P^{2sd-k\ell+\varepsilon}.$$

*Proof.* Let $\boldsymbol{\alpha} \in \mathfrak{M}$. For each $\mathbf{j}$ with $|\mathbf{j}| = k$, we apply Dirichlet's Theorem to obtain integers $q_{\mathbf{j}}$, $a_{\mathbf{j}}$ with $(q_{\mathbf{j}}, a_{\mathbf{j}}) = 1$ such that $|q_{\mathbf{j}}\alpha_{\mathbf{j}} - a_{\mathbf{j}}| \le P^{1-k}$ and $1 \le q_{\mathbf{j}} \le P^{k-1}$. Then by applying Lemma 2.2 together with a familiar transference principle (see for example Vaughan [30], Exercise 2.2), we obtain

$$f(\boldsymbol{\alpha}) \ll P^{d+\varepsilon}\big((q_{\mathbf{j}} + |q_{\mathbf{j}}\alpha_{\mathbf{j}} - a_{\mathbf{j}}|P^k)^{-1} + P^{-1} + (q_{\mathbf{j}} + |q_{\mathbf{j}}\alpha_{\mathbf{j}} - a_{\mathbf{j}}|P^k)P^{-k}\big)^{2^{1-k}}.$$

If $q_{\mathbf{j}} > P$ for some $\mathbf{j}$, then we find that $f(\boldsymbol{\alpha}) \ll P^{d-2^{1-k}+\varepsilon}$ for all $\varepsilon > 0$, whence $\boldsymbol{\alpha} \in \mathfrak{m}$. Thus we may suppose that $q_{\mathbf{j}} \le P$ for all $\mathbf{j}$, and hence that

$$f(\boldsymbol{\alpha}) \ll P^{d+\varepsilon}(q_{\mathbf{j}} + |q_{\mathbf{j}}\alpha_{\mathbf{j}} - a_{\mathbf{j}}|P^k)^{-2^{1-k}} \tag{2.4}$$

for every $\mathbf{j}$. We now let $\mathfrak{M}_{\mathbf{q},\mathbf{a}}$ denote the set of $\boldsymbol{\alpha} \in \mathfrak{M}$ for which (2.4) holds, and set $\mathcal{A}_{\mathbf{q}} = \prod_{|\mathbf{j}|=k}[1, q_{\mathbf{j}}]$. On writing $s = u + \ell$, where $u \ge 2^{k-2}\ell$, we therefore have

$$\int_{\mathfrak{M}} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \ll P^{2ud+\varepsilon} \sum_{\mathbf{q}\in[1,P]^\ell} \sum_{\mathbf{a}\in\mathcal{A}_{\mathbf{q}}} \int_{\mathfrak{M}_{\mathbf{q},\mathbf{a}}} \frac{|f(\boldsymbol{\alpha})|^{2\ell} d\boldsymbol{\alpha}}{\prod_{|\mathbf{j}|=k}(q_{\mathbf{j}} + |q_{\mathbf{j}}\alpha_{\mathbf{j}} - a_{\mathbf{j}}|P^k)}.$$

Now put $P_{\mathbf{j}} = q_{\mathbf{j}}^{-1}P^{1-k}$,

$$\widetilde{\mathbf{q}} = \prod_{|\mathbf{j}|=k} q_{\mathbf{j}}, \quad S(q,r) = \sum_{a=1}^{q} e(ar/q), \quad \text{and} \quad \sigma_{\mathbf{j}}(\mathbf{x}) = \sum_{i=1}^{\ell}(\mathbf{x}_i^{\mathbf{j}} - \mathbf{x}_{\ell+i}^{\mathbf{j}}).$$

Then by applying the technique of Brüdern [7], Lemma 2, we find that

$$\int_{\mathfrak{M}} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \ll P^{2ud+\varepsilon} \sum_{\mathbf{q}\in[1,P]^\ell} \widetilde{\mathbf{q}}^{-1} \sum_{\mathbf{x}\in[1,P]^{2\ell}} \prod_{|\mathbf{j}|=k} S(q_{\mathbf{j}}, \sigma_{\mathbf{j}}(\mathbf{x})) \int_{-P_{\mathbf{j}}}^{P_{\mathbf{j}}} \frac{d\beta_{\mathbf{j}}}{1+|\beta_{\mathbf{j}}|P^k}$$
$$\ll P^{2ud-k\ell+\varepsilon} \sum_{\mathbf{q}\in[1,P]^\ell} M_{\mathbf{q}}(P), \tag{2.5}$$

where $M_{\mathbf{q}}(P)$ is the number of solutions of the system of congruences

$$\sigma_{\mathbf{j}}(\mathbf{x}) \equiv 0 \pmod{q_{\mathbf{j}}} \qquad (|\mathbf{j}| = k) \tag{2.6}$$

with $\mathbf{x} \in [1,P]^{2\ell}$. By a divisor estimate, a given solution of (2.6) can have $\sigma_{\mathbf{j}}(\mathbf{x}) \neq 0$ for only $O(P^\varepsilon)$ values of $\mathbf{q}_{\mathbf{j}}$. When $\mathcal{T} \subseteq \mathcal{J} = \{\mathbf{j} : |\mathbf{j}| = k\}$, write $N(P;\mathcal{T})$ for the number of $\mathbf{x} \in [1,P]^{2\ell}$ satisfying $\sigma_{\mathbf{j}}(\mathbf{x}) = 0$ for all $\mathbf{j} \in \mathcal{T}$. Then since $\ell \geq |\mathcal{T}|$, we have by Lemma 2.3 that $N(P;\mathcal{T}) \ll P^{2\ell d-|\mathcal{T}|}$ and hence

$$\sum_{\mathbf{q}\in[1,P]^\ell} M_{\mathbf{q}}(P) \ll \sum_{\mathcal{T}\subseteq\mathcal{J}} \sum_{\mathbf{q}\in[1,P]^{|\mathcal{T}|}} P^\varepsilon N(P;\mathcal{T}) \ll P^{2\ell d+\varepsilon}.$$

The lemma now follows on substituting this estimate in (2.5). □

## 3. Mean values via Hua-type iteration

We now aim to describe methods for relating $I_{s,k,d}(P)$ with fixed $k$ to mean values associated with smaller $s$ and $d$. We call $\Delta_{s,k,d}$ an admissible exponent for $I_{s,k,d}(P)$ if for all $\varepsilon > 0$ one has the estimate

$$I_{s,k,d}(P) \ll P^{2sd-k\ell+\Delta_{s,k,d}+\varepsilon}. \tag{3.1}$$

We also find it useful to call $\eta_{s,k,d}$ an admissible exponent for $J_{s,k,d}(P)$ if one has

$$J_{s,k,d}(P) \ll P^{2sd-K+\eta_{s,k,d}+\varepsilon} \tag{3.2}$$

for all $\varepsilon > 0$, where, according to [24], Lemma 2.1,

$$K = \frac{dk}{d+1}\binom{k+d}{d}$$

denotes the total degree of the system (1.7). In what follows, we frequently refer to an exponent simply as admissible, with the appropriate context (3.1) or (3.2) indicated by our choice of notation.

To get the iteration started for a given $k$ and $d$, we may apply Lemma 2.3 with $s = \ell$ to conclude that $\Delta_{s,k,d} = (k-1)\ell$ is admissible, but it is frequently superior to appeal to the following observation, in which we make use of available lower-dimensional estimates for systems of the shape (1.4) and (1.7).

**Lemma 3.1.** *Whenever $1 \leq m \leq d-1$, one has*

$$I_{s,k,d}(P) \ll I_{s,k,d-m}(P) J_{s,k,m}(P).$$

*Proof.* For $1 \leq i \leq 2s$, we write $\mathbf{x}_i = (\mathbf{y}_i, \mathbf{z}_i)$, where $\mathbf{y}_i \in \mathbb{Z}^m$ and $\mathbf{z}_i \in \mathbb{Z}^{d-m}$. By first considering the subsystem of equations of (1.4) with $j_1 = \cdots = j_m = 0$, we see that that the number of possibilities for $\mathbf{z}_1, \ldots, \mathbf{z}_{2s}$ is at most $I_{s,k,d-m}(P)$. We now fix any such choice of $\mathbf{z}$, write $z_i$ for the first component of $\mathbf{z}_i$, and consider the subsystem of (1.4) with $j_1 + \cdots + j_m > 0$ and $j_1 + \cdots + j_{m+1} = k$. We aim to show that the number of choices for $\mathbf{y}$ is bounded by $J_{s,k,m}(P)$. To this end, we now write $\mathbf{j} = (j_1, \ldots, j_m)$ and observe that the number of possibilities for $\mathbf{y}_1, \ldots, \mathbf{y}_{2s}$ is bounded above by

$$J^*(P; \mathbf{z}) = \int_{[0,1)^r} \prod_{i=1}^{s} g(\boldsymbol{\alpha}; z_i) g(-\boldsymbol{\alpha}; z_{s+i})\, d\boldsymbol{\alpha},$$

where

$$g(\boldsymbol{\alpha}; z) = \sum_{\mathbf{y} \in [1,P]^m} e\bigg( \sum_{1 \le |\mathbf{j}| \le k} \alpha_{\mathbf{j}} \mathbf{y}^{\mathbf{j}} z^{k-|\mathbf{j}|} \bigg),$$

and $r = \binom{k+m}{m} - 1$. By Hölder's inequality, one has

$$J^*(P; \mathbf{z}) \ll \prod_{i=1}^{2s} \bigg( \int_{[0,1)^r} |g(\boldsymbol{\alpha}; z_i)|^{2s}\, d\boldsymbol{\alpha} \bigg)^{1/(2s)},$$

and on making the change of variable $\beta_{\mathbf{j}} = z_i^{k-|\mathbf{j}|} \alpha_{\mathbf{j}}$, we deduce that

$$J^*(P; \mathbf{z}) \ll \prod_{i=1}^{2s} \bigg( z_i^{-\kappa} \int_{\mathcal{U}_i} |g(\boldsymbol{\beta}; 1)|^{2s}\, d\boldsymbol{\beta} \bigg)^{1/(2s)},$$

where

$$\mathcal{U}_i = \prod_{1 \le |\mathbf{j}| \le k} z_i^{k-|\mathbf{j}|}[0,1) \qquad \text{and} \qquad \kappa = \sum_{1 \le |\mathbf{j}| \le k} (k - |\mathbf{j}|) = \binom{k+m}{m+1} - k.$$

Now by periodicity we finally obtain

$$J^*(P; \mathbf{z}) \ll \prod_{i=1}^{2s} \bigg( \int_{[0,1)^r} |g(\boldsymbol{\beta}; 1)|^{2s}\, d\boldsymbol{\beta} \bigg)^{1/(2s)} \ll J_{s,k,m}(P),$$

and this completes the proof. $\square$

The Weyl differencing algorithm developed in Lemma 2.1 allows us to relate $I_{s,k,d}(P)$ to mean values corresponding to smaller values of $s$ in a style reminiscent of Hua's Lemma. The new feature of the multidimensional case is that there are numerous options for creatively choosing the differencing vector to ensure that the difference polynomials are identically zero for a relatively large subset of the indices $\mathbf{j}$. This strategy facilitates relationships with homogeneous subsystems that mimic $I_{s,k,m}(P)$ or $J_{s,k,m}(P)$ for some $m < d$ in a manner similar to Lemma 3.1, and hence we may iterate simultaneously with respect to $s$ and to $d$. If we are able to difference fewer than $k-1$ times, then there is the potential to save more per variable in the early stages of the iteration than the factor of $P^{2^{1-k}}$ that would stem from Lemma 2.2. We highlight the following simple but effective procedure, in which we spread the differences out over as many variables as possible to maximize the homogeneity of the resulting Hua-type system.

**Lemma 3.2.** *Suppose that $d \ge 3$, $1 \le m \le d-2$, and $m < j \le \min(d, k-1)$. Then one has*

$$I_{s+2^{j-1},k,d}(P) \ll P^{d2^j - 1} I_{s,k,d}(P) + P^{d(2^j-1)-j+\varepsilon} I_{s,k,d-m}(P) J_{s,k,m}(P).$$

*Proof.* Applying Lemma 2.1 with $\mathbf{i} = \mathbf{e}_1 + \cdots + \mathbf{e}_j$ shows that

$$I_{s+2^{j-1},k,d}(P) \ll P^{d(2^j-1)-j} U_{s,k,d}(P), \tag{3.3}$$

where $U_{s,k,d}(P)$ is the number of solutions of the system

$$\begin{aligned} h_1 \cdots h_j p_{\mathbf{ij}}(\mathbf{x}; \mathbf{h}) &= \sum_{i=1}^{s} (\mathbf{y}_i^{\mathbf{j}} - \mathbf{y}_{s+i}^{\mathbf{j}}), \qquad \mathbf{j} \in \mathcal{J}(\mathbf{i}) \\ 0 &= \sum_{i=1}^{s} (\mathbf{y}_i^{\mathbf{j}} - \mathbf{y}_{s+i}^{\mathbf{j}}), \qquad \mathbf{j} \notin \mathcal{J}(\mathbf{i}) \end{aligned} \tag{3.4}$$

with $\mathbf{h} \in (-P, P)^j$, $\mathbf{x} \in [1, P]^d$, and $\mathbf{y}_1, \ldots, \mathbf{y}_{2s} \in [1, P]^d$. We observe that if

$$\mathbf{j} = \mathbf{e}_1 + \cdots + \mathbf{e}_j + (k-j)\mathbf{e}_l \tag{3.5}$$

for some $l$ with $1 \le l \le d$ then $\mathbf{j} \in \mathcal{J}(\mathbf{i})$. Moreover, for $\mathbf{j}$ of the shape (3.5) and a given $\mathbf{h}$, one has $p_{\mathbf{ij}}(\mathbf{x}; \mathbf{h}) = p(x_l; \mathbf{h})$, where $p(x; \mathbf{h})$ is a polynomial in one variable of degree $k - j$ with integer coefficients. We classify solutions counted by (3.4) according to whether

$$h_1 \cdots h_j p(x_1; \mathbf{h}) \cdots p(x_d; \mathbf{h}) = 0. \tag{3.6}$$

Consider first the solutions for which (3.6) holds. Clearly, the number of choices for $h_1, \ldots, h_j$ and $x_1, \ldots, x_d$ in this situation is $O(P^{j+d-1})$, and for any such choice the number of possibilities for $\mathbf{y}_1, \ldots, \mathbf{y}_{2s}$ is bounded above by $I_{s,k,d}(P)$. Next we consider solutions in which (3.6) does not hold. We write $\mathbf{y}_i = (\mathbf{z}_i, \mathbf{w}_i)$, where $\mathbf{z}_i \in \mathbb{Z}^m$ and $\mathbf{w}_i \in \mathbb{Z}^{d-m}$. Here the indices $\mathbf{j}$ for which $j_1 = \cdots = j_m = 0$ all satisfy $\mathbf{j} \notin \mathcal{J}(\mathbf{i})$, whence the second subsystem in (3.4) shows that the number of possibilities for $\mathbf{w}_1, \ldots, \mathbf{w}_{2s}$ is at most $I_{s,k,d-m}(P)$. We now fix any such choice of $\mathbf{w}$ and observe that since $j > m$ the indices $\mathbf{j}$ with $j_{m+1} = \cdots = j_{d-1} = 0$ also satisfy $\mathbf{j} \notin \mathcal{J}(\mathbf{i})$. Hence the proof of Lemma 3.1 shows that, for fixed $\mathbf{w}$, the number of possibilities for $\mathbf{z}$ satisfying the second subsystem of (3.4) is at most $J_{s,k,m}(P)$. Now for fixed $\mathbf{w}$ and $\mathbf{z}$, a consideration of the subsystem of (3.4) for which $\mathbf{j}$ has the special shape (3.5), together with an elementary estimate for the divisor function, shows that $\mathbf{h}$ and $p(x_1; \mathbf{h}), \ldots, p(x_d; \mathbf{h})$ are determined to $O(P^\varepsilon)$. Since for a given $h$ and $n$ the polynomial $p(x_l; \mathbf{h}) - n$ has $O(1)$ roots, it now follows that

$$U_{s,k,d}(P) \ll P^{j+d-1} I_{s,k,d}(P) + P^\varepsilon I_{s,k,d-m}(P) J_{s,k,m}(P),$$

and the proof is completed on substituting this relation into (3.3). □

There are some intrinsic limitations to arguments of the above type resulting from the fact that the total degree of the lower-dimensional subsystems considered is less than that of the original system. In particular, it is impossible to achieve small values of $\Delta_{s,k,d}$ using these processes alone, and hence their utility is confined to relatively small values of $s$. For larger values of $s$, we make use of a Hardy-Littlewood dissection based on Lemmas 2.2 and 2.4, which eventually produces the full savings in (3.1). This final ingredient in the iteration is recorded in the following lemma.

**Lemma 3.3.** *Suppose that the exponent $\Delta_{s,k,d}$ is admissible and that $s + t \ge (2^{k-2} + 1)\ell$. Then*

$$\Delta_{s+t,k,d} = \max(0, \Delta_{s,k,d} - t2^{2-k})$$

*is admissible. In particular, if $t \ge 2^{k-2}\Delta_{s,k,d}$, then $\Delta_{s+t,k,d} = 0$ is admissible.*

*Proof.* We apply a Hardy-Littlewood dissection, recalling the definitions of $\mathfrak{m}$ and $\mathfrak{M}$ from Section 2. If $\Delta_{s,k,d}$ is admissible, then Lemma 2.2 gives

$$\int_{\mathfrak{m}} |f(\boldsymbol{\alpha})|^{2s+2t}\, d\boldsymbol{\alpha} \ll \big(\sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})|\big)^{2t} \int_{[0,1)^\ell} |f(\boldsymbol{\alpha})|^{2s}\, d\boldsymbol{\alpha} \ll P^{(2s+2t)d-k\ell+\Delta_{s,k,d}-t2^{2-k}+\varepsilon},$$

while Lemma 2.4 yields

$$\int_{\mathfrak{M}} |f(\boldsymbol{\alpha})|^{2s+2t}\, d\boldsymbol{\alpha} \ll P^{(2s+2t)d-k\ell+\varepsilon},$$

and the result follows. □

We now record the number of variables needed to obtain $\Delta_{s,k,d} = 0$ for the pairs $(k, d)$ appearing in Theorems 1.1, 1.2, and 1.3. Here we find it convenient to define $s_0(k, d)$ to be the smallest integer $s$ for which $\Delta_{s,k,d} = 0$. The following lemma provides a general argument for handling cubic systems of arbitrary dimension.

**Lemma 3.4.** *Whenever $d \geq 2$, one has*

$$s_0(3, d) \leq d^3 + 3d^2 - 10d + 14.$$

*Proof.* We proceed by induction on $d$. When $d = 2$, we recall from [20], Theorem 7, the estimate

$$J_{s,3,1}(P) \ll P^{2s-6+\varepsilon} \qquad (s \geq 8) \tag{3.7}$$

and apply this in combination with Lemma 3.1 and Hua's Lemma ([30], Lemma 2.5) to obtain $\Delta_{8,3,2} = 3$. From here we proceed directly to Lemma 3.3 with $t = 6$ to obtain $\Delta_{14,3,2} = 0$, whence $s_0(3, 2) \leq 14$. Now let $s = d^3 + 3d^2 - 10d + 14$, and suppose that $s_0(3, d) \leq s$ for some $d \geq 2$, so that $\Delta_{s,3,d} = 0$ is admissible. Combining this with (3.7) in Lemma 3.1 shows that

$$\Delta_{s,3,d+1} = 3\binom{d+3}{3} - 3\binom{d+2}{3} - 6 = \frac{3}{2}d^2 + \frac{9}{2}d - 3$$

is admissible. Hence Lemma 3.3 with $t = 2\Delta_{s,3,d+1}$ shows that $\Delta_{s+t,3,d+1} = 0$ is admissible, and a simple calculation confirms that

$$s + t = (d + 1)^3 + 3(d + 1)^2 - 10(d + 1) + 14,$$

which establishes the required bound for $s_0(3, d + 1)$. □

Alternatively, we note that Lemma 2.3 applied to the full system (1.4) gives $\Delta_{\ell,3,d} = 2\ell$, where $\ell = \ell(3, d) = \binom{d+2}{3}$. Hence it follows from Lemma 3.3 with $t = 4\ell$ that $\Delta_{5\ell,3,d} = 0$, whence

$$s_0(3, d) \leq 5\binom{d+2}{3} = \frac{5}{6}\Big(d^3 + 3d^2 + 2d\Big), \tag{3.8}$$

and this proves to be superior to Lemma 3.4 whenever $d \geq 7$.

When $d = 2$ we instead apply Wooley [40], Theorem 1.1, to obtain $\eta_{k(k+1),k,1} = \Delta_{k(k+1),k,1} = 0$, which when inserted in Lemma 3.1 yields

$$\Delta_{k(k+1),k,2} = k(k + 1) - \tfrac{1}{2}k(k + 1) - k = \tfrac{1}{2}k(k - 1). \tag{3.9}$$

Applying Lemma 3.3 with $t = 2^{k-3}k(k - 1)$ then gives

$$s_0(k, 2) \leq k(k - 1)2^{k-3} + k(k + 1). \tag{3.10}$$

The proof of the following lemma illustrates the wider variety of strategies available when $k \geq 4$ and $d \geq 3$. The main point is that Lemma 3.2 now offers a potential advantage over Lemma 3.3 in the early stages of the iteration provided that $j < k-1$.

**Lemma 3.5.** *One has*

$$s_0(4,3) \leq 146, \quad s_0(4,4) \leq 404, \quad \textit{and} \quad s_0(5,3) \leq 546.$$

*Proof.* (i) When $(k,d) = (4,3)$, we apply (3.9) and [40] to obtain the admissible exponents $\Delta_{20,4,2} = 6$ and $\eta_{20,4,1} = 0$, and inserting these into Lemma 3.1 gives $\Delta_{20,4,3} = 60-14-10 = 36$. Next, we apply Lemma 3.2 with $j = 2$ and $m = 1$ to get

$$I_{s+2,4,3}(P) \ll P^{11} I_{s,4,3}(P) + P^{7+\varepsilon} I_{s,4,2}(P) J_{s,4,1}(P).$$

Here the results of (3.9), (3.10), and [40] show that the first term on the right-hand side dominates when $20 \leq s \leq 36$ with the currently available estimates for $I_{s,4,3}(P)$. Thus after nine applications of this inequality we obtain $\Delta_{38,4,3} = 27$. Finally, an application of Lemma 3.3 with $t = 4 \cdot 27 = 108$ delivers $\Delta_{146,4,3} = 0$.

(ii) The case $(k,d) = (4,4)$ is similar. By (i) and [40] we have $\Delta_{20,4,3} = 36$ and $\eta_{20,4,1} = 0$, and hence Lemma 3.1 gives $\Delta_{20,4,4} = 140 - 24 - 10 = 106$. Next, we apply Lemma 3.2 with $j = 2$ and $m = 1$ to get

$$I_{s+2,4,4}(P) \ll P^{15} I_{s,4,4}(P) + P^{10+\varepsilon} I_{s,4,3}(P) J_{s,4,1}(P),$$

and the exponents obtained from (i), in combination with [40], show that the first term on the right dominates when $20 \leq s \leq 58$. Thus after 20 iterations we obtain $\Delta_{60,4,4} = 86$, and an application of Lemma 3.3 with $t = 4 \cdot 86 = 344$ delivers $\Delta_{404,4,4} = 0$.

(iii) Finally, let $(k,d) = (5,3)$. On recalling (3.9) and [40], we find that Lemma 3.1 gives $\Delta_{30,5,3} = 105 - 20 - 15 = 70$. Six applications of Lemma 3.2 with $j = 2$ and $m = 1$ in the style of (i) and (ii) above then yields $\Delta_{42,5,3} = 64$. Next we apply Lemma 3.2 with $j = 3$ and $m = 2$ to get

$$I_{s+4,5,3}(P) \ll P^{23} I_{s,5,3}(P) + P^{18+\varepsilon} I_{s,5,2}(P) J_{s,5,1}(P),$$

and after two applications of this relation we obtain $\Delta_{50,5,3} = 62$. Finally, an application of Lemma 3.3 with $t = 8 \cdot 62 = 496$ gives the result. $\square$

## 4. The asymptotic formula

The proofs of our theorems are now accessible. We start by observing that

$$N_{s,k,d}(P) = \int_{[0,1)^\ell} \prod_{i=1}^{s} f_i(\boldsymbol{\alpha})\, d\boldsymbol{\alpha},$$

where

$$f_i(\boldsymbol{\alpha}) = \sum_{\mathbf{x} \in [-P,P]^d} e\Bigg( \sum_{|\mathbf{j}|=k} c_{i\mathbf{j}} \alpha_{\mathbf{j}} \mathbf{x}^{\mathbf{j}} \Bigg).$$

Set $\nu = 1/(2\ell)$ and write $c$ for the least common multiple of the $|c_{i\mathbf{j}}|$ with $1 \leq i \leq s$ and $|\mathbf{j}| = k$. Define the major arc

$$\mathfrak{N}(q,\mathbf{a}) = \{\boldsymbol{\alpha} \in [0,1)^\ell : |q\alpha_{\mathbf{j}} - a_{\mathbf{j}}| \leq P^{1/2-k} \ (|\mathbf{j}| = k)\},$$

and let $\mathfrak{N}$ denote the union of all $\mathfrak{N}(q,\mathbf{a})$ with $\mathbf{a} \in [1,q]^\ell$, $(q,\mathbf{a}) = 1$, and $q \leq cP^{1/2}$. Finally, write $\mathfrak{n} = [0,1)^\ell \setminus \mathfrak{N}$ for the minor arcs. Consider a fixed $i$ with $1 \leq i \leq s$. When

$\boldsymbol{\alpha} \in \mathfrak{n}$, we apply Dirichlet's Theorem for each $\mathbf{j}$ with $|\mathbf{j}| = k$ to obtain integers $b_{\mathbf{j}}$ and $q_{\mathbf{j}} \le P^{k-\nu}$ with $(q_{\mathbf{j}}, b_{\mathbf{j}}) = 1$ such that $|q_{\mathbf{j}} c_{i\mathbf{j}} \alpha_{\mathbf{j}} - b_{\mathbf{j}}| \le P^{\nu - k} \le q_{\mathbf{j}}^{-1}$. If $q_{\mathbf{j}} \le P^{\nu}$ for each $\mathbf{j}$, then on setting $q = c \prod_{|\mathbf{j}|=k} q_{\mathbf{j}} \le cP^{1/2}$ and $a_{\mathbf{j}} = q b_{\mathbf{j}}/(c_{i\mathbf{j}} q_{\mathbf{j}})$, we obtain

$$|\alpha_{\mathbf{j}} - a_{\mathbf{j}}/q| \le (c_{i\mathbf{j}} q_{\mathbf{j}})^{-1} P^{\nu-k} \le q^{-1} P^{1/2-k} \quad (|\mathbf{j}| = k),$$

contradicting the assumption that $\boldsymbol{\alpha} \in \mathfrak{n}$. Therefore we must have $q_{\mathbf{j}} > P^{\nu}$ for some $\mathbf{j}$, and hence Lemma 2.2 yields the estimate

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{n}} |f_i(\boldsymbol{\alpha})| \ll P^{d - 2^{1-k}\nu + \varepsilon}.$$

Upon recalling the definition of $s_0(k,d)$ from Section 3, we find using Hölder's inequality and a change of variables that whenever $s \ge 2s_0(k,d) + 1$ one has

$$\int_{\mathfrak{n}} \prod_{i=1}^{s} |f_i(\boldsymbol{\alpha})| \, d\boldsymbol{\alpha} \ll P^{sd - k\ell - \delta}$$

for some $\delta > 0$. Moreover, the arguments underlying [24], Theorems 1.4 and 6.2 (see also [25], Lemma 5.4) demonstrate that whenever $s \ge k(\ell + 1) + 2$ one has

$$\int_{\mathfrak{N}} \left( \prod_{i=1}^{s} f_i(\boldsymbol{\alpha}) \right) d\boldsymbol{\alpha} = \mathfrak{J}\mathfrak{S} P^{sd - k\ell} + O(P^{sd - k\ell - \delta})$$

for some $\delta > 0$, where $\mathfrak{J}$ and $\mathfrak{S}$ denote the singular integral and singular series defined by

$$\mathfrak{J} = \int_{\mathbb{R}^\ell} \int_{[-1,1]^{sd}} e\biggl( \sum_{|\mathbf{j}|=k} \beta_{\mathbf{j}} (c_{1\mathbf{j}} \boldsymbol{\gamma}_1^{\mathbf{j}} + \cdots + c_{s\mathbf{j}} \boldsymbol{\gamma}_s^{\mathbf{j}}) \biggr) d\boldsymbol{\gamma} \, d\boldsymbol{\beta} \tag{4.1}$$

and

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{a} \in [1,q]^\ell \\ (q,\mathbf{a})=1}} q^{-sd} \prod_{i=1}^{s} \sum_{\mathbf{x} \in [1,q]^d} e\biggl( q^{-1} \sum_{|\mathbf{j}|=k} c_{i\mathbf{j}} a_{\mathbf{j}} \mathbf{x}^{\mathbf{j}} \biggr). \tag{4.2}$$

The positivity of $\mathfrak{J}$ and $\mathfrak{S}$ under the hypothesis of non-singular local solutions follows from standard arguments (see for example [25], §6), and the proof of Theorem 1.3 is now completed by recalling Lemma 3.5. Similarly, Theorem 1.2 now follows from Lemma 3.4, (3.8), and (3.10). Finally, when $k = 3$ and $d = 2$, we deduce from the argument of [22], Lemma 5.1, that $\mathfrak{J}\mathfrak{S} > 0$ unconditionally whenever $s \ge 14$, and the conclusion of Theorem 1.1 therefore follows from Lemma 3.4.

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